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Two-Body Problem in the Hyperbolic-Astrodynamical-Coördinate Mesh

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The plane-polar coördinates, not being the natural choice for setting up two-body problem, leads to introduction of the elliptic-astrodynamical-coördinate mesh formulated to represent orbital problems involving 2 bodies. The original problem consisting of (6+6) degrees-of-freedom (3 translational + 3 rotational for each of the two bodies) is reduced to 3 by neglecting body structures and writing the equations of motion in the center-of-mass frame-of-reference (which is inertial in the absence of external forces). Absence of external torques ensures conservation of angular momentum, having both magnitude and direction (a vector quantity). Orbits are located in a plane, as angular-momentum vector (fixed in space) is, always, normal to orbital plane. This makes the problem 2 dimensional. Elliptic-astrodynamical-coördinate mesh is the underlying structure to write the orbital equation of motion. This makes the formulation one degree-of-freedom problem. The equation of motion is shown to have Kepler's equation as a particular solution. The orbits of the two-body problems are ellipses, when the system is bound and the total energy of the system is negative. The system is free when the total energy either vanishes (parabolic orbit) or is positive (hyperbolic orbit). Example of the later is alpha-particle scattering, which led to the discovery of nucleus (Rutherford scattering), where the orbit of alpha particle is hyperbola. Mathematical modeling of two-body problem, resulting in hyperbolic orbits, becomes simpler when the authors use hyperbolic-astrodynamical-coördinate mesh (κ, H, z) — κ is hyperbolic-shape coördinate, $\frac{1}{ae}\tanh^{-1}\eta$ (a is semi-major axis of the hyperbola, e eccentricity, which is greater than unity for hyperbola and $\eta=\sqrt{e^2-1}$), H is hyperbolic-eccentric anomaly and z is the coördinate representing direction of angular momentum. Scale factors are computed as $h_\kappa = \frac{A}{B}$ ($A = e\cosh H - 1$, $B = \frac{\kappa}{a}$); $h_{\mathrm{H}} = a\sqrt{1+e^2\cosh^2 H}$; $h_z = 1$. Lagrangian of the two-body problem comes out to be $\mathcal{L}=\frac{mMa^2(e^2\cosh^2 H-1)}{2(m+M)}\dot{H}^2+\frac{GmM}{a(e\cosh H-1)}$; m and M are masses of lighter and heavier bodies of the pair, respectively and G was the universal gravitational constant. The canonical momenta are computed as $p_\kappa = \frac{\partial L}{\partial\dot{\kappa}}$ = constant, $p_{\mathrm{H}} = \frac{\partial L}{\partial \dot{H}} = \frac{mMa^2(e^2\cosh^2 H-1)}{m+M}\dot{H}$. Hamiltonian of this system is expressed as $\mathcal{H}=\frac{1}{a(e\cosh H-1)}\left[\frac{(m+M)p_{\mathrm{H}^2}}{2mMa^2(1+e\cosh H)}+GmM\right]$. This formulation has 3 constants of motion (instead of the customary 2, which are characteristic of plane-polar-coördinate formulation found in text books), namely, hyperbolic-shape coördinate, κ, canonical momentum corresponding to hyperbolic-shape coördinate, p_κ, and energy, $\mathcal{E}$ (the coördinate transformations do not depend on time, hence the hamiltonian function represents energy of the system; no explicit dependence on time, t, conserved the energy). This formulation might find use in *Airspacecraft of the Third Millennium*, where choice of the hyperbolic orbits, instead of the elliptical orbits, could reduce travel time between two cities, say, Karachi and New York, significantly.

Keywords: Constants of motion • Elliptic-astrodynamical-coördinate mesh • Equation of motion • Hyperbolic orbits

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