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SOLUTIONS OF THE EINSTEIN EQUATION IN COSMOLOGY

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ABSTACT

Solutions of the Einstein equation governing the dynamical evolution of the universe are obtained in a simple way for matter-dominated era in the standard model. Physical implications of specific parameter values are discussed.

1. INTRODUCTION

If the geometry of spacetime is Riemannian and our observed universe, probably, satisfies the conditions of homogeneity, isotropy and negligible pressure to a good approximation, the standard model can be used to describe the dynamical evolution of our universe. Solutions for the scale factor $a(t)$ in the standard model are familiar in textbooks, but often appear there in an abstract presentation, which fails to render the full range and the continuity of the solutions and the simple physical content of the results.

In this paper solutions of Einstein equation in the matter-dominated era are obtained for zero-pressure Friedmann universe with the cosmological constant set equal to zero. Implications of specific parameter values are discussed. Some consequences of the expansion of the universe are, also, described.

2. THE STANDARD MODEL

If the spacetime has a Riemannian metric (a fairly weak assumption) and if the three-space is homogeneous and isotropic (the cosmological principle), then the line element dl in any three-space may be written as (Peebles, 1971; Weinberg, 1972):

(1) $$dl^2 = a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + d\varphi^2 \sin^2\varphi)]$$

where $a(t)$ is an unknown function of time, usually called the scale factor, k is a constant, which by suitable choice of units for r can be chosen to have the value +1, 0, –1, (r,θ,φ) are spherical-

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polar coordinates with θ as the polar angle. dl is measured in cm. The metric (1) is known in cosmology as the *Robertson-Walker Metric*.

The universe is *open*, *flat* or *closed* if k = +1, 0, –1, respectively. The three-space with metric (1) has intrinsic Gaussian curvature (units of cm^{-2})

(2) $$K(t) = ka^{-2}(t)$$

K is uniform over the three-space but varies in time. If k is positive, $K^{-1/2}$ is called radius of curvature or radius of the universe. The universe is, then, closed, having a finite volume

(3) $$V = 2\pi^2 K^{-3/2}$$

Closure by itself does not tell whether or not the universe shall recollapse. Felton and Isaacman (1986) have shown that in relativistic cosmology some open models recollapse and closed ones do not.

In the standard cosmology universe is a perfect gas and the system is isolated. It starts at infinite temperature(big bang) and cools with expansion. The energy density, ρ, pressure, *p*, and the scale factor, *a*, are related by

(4) $$\frac{dp}{dt} + \frac{3(\rho + p)}{a}\frac{da}{dt} = 0$$

In the radiation-dominated era (early universe) particles are moving relativistically. The pressure is non-negligible. For the very-early universe $\rho = 3p$ and so eq. (4), then, gives

(5) $$\rho \propto a^{-4}$$

Although, the number density of photons is constant, as the universe expands the photons are red shifted and we, eventually, have a matter-dominated universe. In the mater-dominated era (the present era) particles are moving non-relativistically. The pressure is negligible. Eq. (4), then, gives

(6) $$\rho \propto a^{-3}$$

Taking the cosmological constant, $\lambda = 0$, pressure, $p = 0$, the dynamical equation governing the universe during this era is the Einstein equation

(7) $$\left(\frac{da}{dt}\right)^2 + kc^2 = \frac{8}{3}\pi G\rho a^2$$

where *c* is the velocity of light in free space and *G* is the constant of universal gravitation. The Hubble constant[1], *H* (constant means no variation in space) and density of the universe, ρ are related by

(8) $$q_0 H_0^2 = \frac{4}{3}\pi G\rho_0$$

where $H_0 = \left(\frac{1}{a}\frac{da}{dt}\right)_0$, $q_0 = -\frac{1}{H_0^2}\left(\frac{1}{a}\frac{d^2a}{dt^2}\right)_0$ and ρ_0 are the present values of the *Hubble constant*, the *deceleration parameter* and the *density of our universe*, respectively. The subscripts are dropped in the rest of the discussion. Using values of H and q in (8), one gets

(9) $$-2a\frac{d^2a}{dt^2} = \frac{8}{3}\pi G\rho a^2$$

Comparing (7) and (9)

(10) $$\left(\frac{da}{dt}\right)^2 + 2a\frac{d^2a}{dt^2} + kc^2 = 0$$

This equation is to be solved for various values of k in the following section.

2. SOLUTIONS OF THE EINSTEIN EQUATION

Substituting $\frac{d^2a}{dt^2} = \frac{d}{dt}\left(\frac{da}{dt}\right) = \frac{d\alpha}{dt} = \alpha\frac{d\alpha}{dt}\frac{dt}{da} = \alpha\frac{d\alpha}{da} = \frac{1}{2}\frac{d}{da}(\alpha^2) = \frac{1}{2}\frac{d\in}{da}$, in eq. (10), one has $(\alpha = \frac{da}{dt}, \alpha^2 = \in)$,

(11) $$a\frac{d\in}{da} = -(kc^2 + \in)$$

a) Flat Universe ($k = 0$)

For $k = 0$, eq. (11), immediately, gives

$$\ln(\in a) = \ln(P^2)$$

where $\ln(P^2)$ is a constant of integration. Recalling that $\in = \alpha^2$, this becomes[2]

(12) $$\sqrt{a}\,da = Pdt$$

or

$$\frac{2}{3}a^{3/2} = Pt + Q$$

where Q is another constant of integration. The most general solution is, therefore

(13) $$a(t) = (A_1 t + B_1)^{2/3}$$

with $A_1 = \frac{3}{2}P$, $B_1 = \frac{3}{2}Q$. The usual result $a(t) \propto t^{2/3}$ follows upon taking $B = 0$.

b) Open Universe ($k = -1$)

For $k = -1$, eq. (11) becomes

$$\frac{da}{a} = \frac{d\in}{c^2 - \in}$$

Integrating and rearranging

$$\in = \left(\frac{da}{dt}\right)^2 = c^2 - \frac{1}{A_2 a}$$

where A_2 is a constant of integration. This can be easily integrated if written in the form[2]

(14)
$$\frac{da}{\sqrt{c^2 - (A_2 a)^{-1}}} = dt$$

Using the substitution, $z^2 = (A_2 a)^{-1}$ and integrating the solution comes out to

(15)
$$\Omega\forall + \frac{\ln(\Omega + \forall)}{A_2 c^3} = t + B_2$$

where B_2 is another constant of integration, and $\Omega = c\sqrt{A_2 a}$, $\forall = \sqrt{c^2 A_2 a - 1}$.

c) Closed Universe ($k = +1$)

For $k = +1$, eq. (11) becomes

$$\frac{da}{a} = -\frac{d\in}{c^2 + \in}$$

straightforward integration yields

$$\in = \left(\frac{da}{dt}\right)^2 = \frac{1}{A_3 a} - c^2$$

where A_3 is a constant of integration. For the present epoch this can be written as[2]

(16)
$$\frac{da}{\sqrt{(A_3 a)^{-1} - c^2}} = dt$$

Again substituting $z^2 = (A_3 a)^{-1}$ and integrating he solution comes out to

(17)
$$\sqrt{A_3 a(1 - c^2 A_3 a)} + \frac{1}{c}\sec^{-1}\frac{1}{c\sqrt{A_3 a}} + A_3 c^2 (t + B_3) = 0$$

where B_3 is another constant of integration.

4. IMPLICATIONS OF SPECIFIC PARAMETER VALUES

The parameters, which can be measured using astronomical techniques are (Misner, Thorne and Wheeler, 1973):

a)	Hubble Constant [Hubble-Expansion Rate (today)]	$H_0 = \frac{\alpha(t_0)}{a(t_0)}$
b)	Deceleration Parameter	$q_0 = -\frac{1}{H_0^2}\left(\frac{d\alpha/dt}{a}\right)_0$
c)	Density Parameter	$\sigma_0 = \frac{4}{3}\pi\rho_{m0}\frac{1}{H_0^2}$

The quantity ρ_{m0} is explained below. The relativity parameters of interest in the standard model are:

a)	Matter Density (today)	ρ_{m0}
b)	Curvature of Hypersurface of Homogeneity (today)	k/a_0^2
c)	Cosmological Constant	λ
d)	Radiation Density (today) (ignored because it is unimportant for the present dynamics of the universe)	ρ_{r0}

Values of the deceleration parameter and the mater density determine whether the universe is closed, open or flat.

a) If $q_0 > 1/2$, we have

$$\rho_{m0} > \rho_{critical} = \frac{3H_0^2}{8\pi G}$$

This implies that $k > 0$, *i. e.*, our universe is *closed*. The universe shall expand initially. At a certain time q_0 becomes infinity, the expansion stops and the universe starts recontracting.

b) If $q_0 = 1/2$, we have

$$\rho_{m0} = \rho_{critical} = \frac{3H_0^2}{8\pi G}$$

This implies that $k = 0$, *i. e.*, our universe is *flat*. Under these conditions the universe shall expand forever.

c) If $q_0 < 1/2$, we have

$$\rho_{m0} < \rho_{critical} = \frac{3H_0^2}{8\pi G}$$

This implies that $k < 0$, *i. e.*, our universe is *open*. An open universe shall expand forever.

Current estimates suggest that $H_0 = 42\ km\,s^{-1} Mpc^{-1}$, and the age of the universe is 15×10^9 *years* , giving the possibility that the geometry of spacetime can be exactly flat according to this time-scale test, and the cosmological density is just the value needed to close the universe. If so, this providers the astronomical justification for hot-dark mater making up 99% of the mass of the universe, with the well-known connection to particle physics (Kim, 1984).

NOTES AND REFERENCES

[1] Since *H* varies with time some authors prefer to use the terminology *Hubble Parameter*. See, for example, Felton and Isaacman (1986).

[2] Positive square root of $\in$ is taken because the current value of the Hubble constant is positive. The same applies to equations (14) and (16).

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