Kar. Univ. J. Sc. Vol. **28 (1)**, June 2000, pp. 95-97

THE QUANTUM-MECHANICAL INTERPRETATION OF TIME

SYED ARIF KAMAL[1, 2] and NASEERUDDIN[3]
Departments of Mathematics[2], Physics[3] and Computer Science[2, 3]
University of Karachi, Karachi, Pakistan

ABSTRACT

If time is substituted in the equation of motion in the Heisenberg picture the results obtained may lead to confusions if the equations are not interpreted correctly. Some examples are given.

Keywords: Time, Heisenberg picture, equation of motion

INTRODUCTION

The equation of motion forms a part of every quantum-mechanics course. However, little thought is given to what happens if time is substituted in the equation of motion. Time rate of change of an operator brings out some apparent paradoxes if one does not comprehend thoroughly the different pictures and the various steps involved in obtaining the uncertainty relation (Robertson, 1929). In the following section these points are illustrated.

STATEMENT OF THE PROBLEM

Consider the equation of motion for an operator represented by Ω as viewed in the Heisenberg picture (Schiff, 1968)

(1) $$\frac{d\Omega_{\mathrm{H}}}{dt} = \frac{\partial \Omega_{\mathrm{H}}}{dt} + \frac{2\pi}{ih}\left[\Omega_{\mathrm{H}}, H\right]$$

The subscript H denotes that the operator is represented in the Heisenberg picture, $i = \sqrt{-1}$, whereas h represents Planck's constant, H the hamiltonian operator, t the time and [ ] the commutator. Substituting t in place of Ω, this becomes

(2) $$\frac{dt}{dt} = \frac{\partial t}{dt} + \frac{2\pi}{ih}\left[t, H\right]$$

Since dt/dt and $\partial t/\partial t$ are both equal to unity, one gets

(3) $$\left[t, H\right] = 0$$

On the other hand, if one tries to evaluate this commutator using $ih\frac{\partial}{\partial t}$ as the operator for H, the result is

(4) $$\left[t, H\right] = \frac{h}{2\pi i}$$

This, however, can easily be shown to be incorrect. The operator conjugate to H does not exist. Pauli showed this in the thirties. The essence of the argument is that if $\exists t$ such that $\left[t, H\right] = \frac{h}{2\pi i}$, then the unitary operator $\exp(i\alpha t)$ would generate a continuous shift of energy. But that is inconsistent with H having a lowest eigen-

---

[1]PhD; MA, Johns Hopkins, Baltimore, MD, United States; MS, Indiana, Bloomington, IN, United States; *paper mail***:** Associate Professor, Department of Mathematics, University of Karachi, Karachi 75270, Pakistan; *e-mail:* profdrakamal@gmail.com; *telephone***:** +92 21 9926 1300-15 ext. 2293; *homepage***:** http://www.ngds-ku.org/kamal

value. Moreover, Eq. (4) cannot be used to evaluate the uncertainty product of energy and time because $ih\frac{\partial}{\partial t}$ is not an energy operator. Using the uncertainty relation for two operators (Robertson, 1929)

(5) $$\Delta O_1 \Delta O_2 \geq \frac{1}{2} < [O_1, O_2] >$$

($< [O_1, O_2] >$ represents average of the commutator) and Eq. (3) if one evaluates the uncertainty product of t and E (the symbol E represents energy) one gets

(6) $$\Delta t \Delta E \geq \frac{1}{2} < [t, H] >$$

which seems to contradict the Heisenberg uncertainty relation $\Delta t \Delta E \sim h$ (~ means "of the order of").

These problems can be resolved if one looks closely at the equation of motion and the uncertainty relation used.

A CLOSER LOOK

The contradiction apparent in Eqs. (3) and (4) is resolved if one notes that the equation of motion is obtained under the assumption that H is independent of time (Schiff, 1968). Since H is independent of time, it must commute with t. Therefore, Eq. (3) is correct.

When H is independent of time, energy of the state does not change with time. Such a state is called a stationary state. One might ask whether a stationary state violates the uncertainty principle $\Delta t \Delta E \sim h$. Since energy of the state is known there is no uncertainty in the measurement of energy, *i. e.*, $\Delta E = 0$. However, to make sure that the energy of the state does not change, one must make measurements for an infinite period of time, *i. e.* from $t = -\infty$ to $t = +\infty$. Thus $\Delta t = \infty$, and the uncertainty product again gives $\Delta t \Delta E \sim h$.

The question arises why (6) gives the result $\Delta t \Delta E = 0$. In fact, this conclusion can be drawn only if one does not pay attention to the derivation of (5). Defining (Messiah, 1958)

$$\Delta O_i = \sqrt{\overline{O_i^2} - \overline{O_i}^2}, \ \Theta = O_i - \overline{O_i} \ ; i = 1,2$$

(bar over a variable represents average) one can immediately verify that

(7) $$\Delta O_i = \Delta \Theta_i = \sqrt{\overline{O_i^2}}$$

Assuming that the dynamical state of the system is represented by the ket |*u*> normalized to unity, and applying the Schwarz inequality

(8) $$(\Delta O_1)^2 (\Delta O_2)^2 \equiv \left| < u \middle| \Theta_1^2 \middle| u > \right| \left| < u \middle| \Theta_2^2 \middle| u > \right| \geq \left| < u \middle| \Theta_1 \Theta_2 \middle| u > \right|$$

Separating the hermitian part from the anti-hermitian part in the expression for $\Theta_1 \Theta_2$, one gets

$$\Theta_1 \Theta_2 = \frac{1}{2}\{\Theta_1, \Theta_2\} + \frac{1}{2}[\Theta_1, \Theta_2] = \frac{1}{2}\{\Theta_1, \Theta_2\} + \frac{1}{2}[O_1, O_2]$$

where $\{\Theta_1, \Theta_2\}$ denotes anticommutator. Therefore, (8) becomes

(9) $$\left(\Delta O_1\right)^2\left(\Delta O_2\right)^2 \geq \frac{1}{2}\{\Theta_1,\Theta_2\}^2 + \frac{1}{2}\left[O_1,O_2\right]^2$$

from which (5) follows. Using (3) in (9) one gets

$$\Delta t \Delta E \geq \frac{1}{2} < tH >\sim h$$

Hence, there is no contradiction.

CONCLUSION

Time in the Heisenberg picture needs special discussion. Whenever the equation of motion is discussed the role of time should be explained so that no confusion remains in the relation. Time in quantum theory (especially relativistic quantum mechanics) is a parameter not an *observable* like '*x*' or '*p*'. Discussions like these could stimulate appropriate thinking and critical analysis of the problem (Kamal & Siddiqui, 1986).

ACKNOWLEDGEMENTS

One of the authors (SAK) would like to thank Abdul Khalique Kukaswadia for drawing his attention to this problem.

REFERENCES

Kamal, S. A, & Siddiqui, K. A. (1986). How to develop creative thinking and critical analysis, *Proc. of 2nd Workshop on Teaching of Physics*, Karachi, Pakistan, pp. 51-56, *full text***:** http://www.ngds-ku.org/Papers/C24.pdf

Messiah, A. (1958). *Quantum Mechanics*, vol. 1 (Amsterdam: North-Holland), p. 300

Roberstson, H. P. (1929). *Phys. Rev.* **34**: 164(L)

Schiff, L.I. (1968). *Quantum Mechanics* (Tokyo: McGraw-Hill Kogakusha), 3rd ed., p. 171

Web address of this document (*first author's homepage*)**:** http://www.ngds-ku.org/Papers/J23.pdf
*Abstract***:** http://www.ngds-ku.org/pub/jourabst.htm#J23: