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COVARIANT GENERALIZED COUPLING MODEL OF GLOBAL ELECTROCORTICAL ACTIVITY

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ABSTRACT

Global electrocortical activity of the human brain is modeled as a system of damped harmonic oscillators. The equations are written in a covariant form using tensorial notation. Generalized coupling dependent on both the electrical potentials and their rates of change is introduced in a recently proposed covariant model of global electrocortical activity. First-order shifts in frequencies are obtained in the presence of a weak, time- varying magnetic field. A condition of impedance matching relates the coupling parameters and damping coefficients.

Keywords: Neuromagnetism, magnetoencephalography

INTRODUCTION

Brain is a complex system and brain theory is developing into a central focus among many disciplines. Functioning of the brain has long been prominent among topics of fundamental interests to biologists and physicists alike. Working of the brain presents one of the most challenging problems of neurosciences.

Wright & Kydd (1984) developed a linear model for global electrocortical activity and its control by lateral hypothalamus. They proposed that the properties of the electrocortical waves be clearly distinguished from the microscopic and nonlinear interactions, which underlie them. The model was generalized (Kamal *et al.*, 1989) to include magnetic fields. In this covariant model we have written equations of electrical potential in dendritic trees in the comoving frame of signal.

In this paper we are suggesting a generalized coupling based on both the potentials as well as their rates of change. In the presence of a weak magnetic field varying linearly with time first-order corrections to the EEG frequencies are calculated. An external magnetic field is expected to modify the frequencies in such a way that the shift in EEG frequencies is proportional to the square root of the magnitude of the rate of change of external magnetic vector potential.

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Linear Model of Wright and Kydd

Wright & Kydd (1984) developed a linear model of global electrocortical activity. Their model rests on drastic simplifications and bypasses issues of cell-to-cell coupling, details of anatomy etc. Essential theoretical features of this model may be summarized as:

a. Electrocortical recordings reflect the transformed spatial average of cortical potentials (Elul, 1972).

b. The telencephalon is assumed to be a linear wave medium with regard to the gross wave potentials although the underlying microscopic interactions may be extremely nonlinear.

c. Closed and constant boundary conditions lead the linear waves to generate activity at a large number of resonant modes, each associated with a constant natural frequency.

d. The values for the natural modes of the resonant frequencies are clustered about certain central values (Cramer's Central Limit Theorem).

e. Ascending inhibitory systems act partly to damp resonant activity and partly as a source of noiselike driving signals.

A mass of unit sources coupled to each other may be represented by a set of driven harmonic oscillator equations (j runs from 1 to n subject to the condition that $i \neq j$; n is the number of synaptic connections ~10^{15})

$$\frac{d^2\varphi_\mathrm{i}}{dt_2} + D_\mathrm{i}(t)\frac{d\varphi_\mathrm{i}}{dt} + N_\mathrm{i}^2(t)\varphi_\mathrm{i} = \sum{}_\mathrm{j} K_\mathrm{i}^\mathrm{i}(t)\varphi_\mathrm{j} \tag{1}$$

where φ_i is the electrical potential in a segment of the dendritic tree and t is time as measured in the laboratory frame, $D_\mathrm{i}(t)$, $N_\mathrm{i}(t)$, $K_\mathrm{i}^\mathrm{j}(t)$ are free parameters analogous to damping coefficients, natural frequencies and coupling constants. These parameters are given physiological meaning under these assumptions:

i. All $N_\mathrm{i}(t)$ [$D_\mathrm{i}(t), K_\mathrm{i}^\mathrm{j}(t)$] have a finite variance $\sigma_\mathrm{N}[\sigma_\mathrm{D}, \sigma_\mathrm{K}]$ about a mean $\overline{N}$, $\left[\overline{D}, K\right]$. No particular type of distribution for $D_\mathrm{i}(t)$, $N_\mathrm{i}(t)$, $K_\mathrm{i}^\mathrm{j}(t)$ is assumed.

ii. All $D_\mathrm{i}(t)$, $N_\mathrm{i}(t)$, $K_\mathrm{i}^\mathrm{j}(t)$ are stochastically independent, as each represents processes being perturbed by very complicated nonlinearities in the interactions of the linked oscillatory sources, with diverse input signals.

Under these assumptions regarding the parameters, the Central Limit Theorem of Cramer applies as n tends to be a large number. This is a very reasonable assumption. All $N_\mathrm{i}(t)$, $D_\mathrm{i}(t)$, $K_\mathrm{i}^\mathrm{j}(t)$ may, therefore, be replaced by $\overline{N}$, $\overline{D}$, $\overline{K}$. The model system is equivalent to a linear, time-invariant system while these values remain unchanged. Eq. (1) may be put in another form by defining a new set of variables. Let

$$z_\mathrm{k} = \varphi_\mathrm{k}, \text{ if } k \text{ is an odd number} \tag{2a}$$

(2*b*) $z_k = d\varphi_{k-1}/dt$, otherwise

In terms of these z_k, Eq. (1) may be written as

$$Z\frac{dZ}{dt} = A_{WK}Z \qquad (3)$$

where $Z = [z_k]$; $k = 1, 2, \ldots, m$ is an $m \times 1$ column vector and A_{WK} is the Wright and Kydd's state transition matrix given by

$$A_{WK} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 \\ -N_1^2 & -D_1 & K_1^2 & 0 & K_1^3 & \ldots\ldots & K_1^n & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots\ldots & 0 & 0 \\ K_2^1 & 0 & -N_2^2 & -D_2 & K_2^3 & \ldots\ldots & K_2^n & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 \\ K_3^1 & 0 & K_3^2 & 0 & -N_3^2 & \ldots\ldots & K_3^n & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 \\ \ldots & & & & & & & \ldots \\ \ldots & & & & & & & \ldots \\ K_n^1 & 0 & K_n^2 & 0 & K_n^3 & \ldots\ldots & -N_n^2 & D_n \end{bmatrix} \qquad (4)$$

The elements in Wright and Kydd's state transition matrix donot all have the same dimensions. For example, $-(N_1)^2$ has units of $(\text{time})^{-2}$ whereas $-D_1$ has units of $(\text{time})^{-1}$. In the next section, the system of equations (3) shall be written so that all the elements of A are dimensionless.

Our Covariant Model

In our covariant model (Kamal *et al.*, 1989) equations for the time variation of potential of a dendritic tree were written in the comoving frame of signal. This comoving frame was fixed to the potential wavefront of dendrite. When this equation was transformed into the laboratory frame a magnetic vector potential appeared alongwith the electrostatic potential.

In this comoving frame a mass of unit sources coupled to each other were represented by the following set of equations

$$\frac{d^2\Phi_i}{d\tau^2} + \boldsymbol{D}_i(\tau)\frac{d\Phi_i}{d\tau} + \boldsymbol{N}_i^2(\tau)\Phi_i = \sum{}_j \boldsymbol{K}_i^j(\tau)\Phi_j \qquad (5)$$

where $\boldsymbol{D}_i(\tau)$, $\boldsymbol{N}_i^2(\tau)$ and $\boldsymbol{K}_i^j(\tau)$ were 4×4 matrices with eigenvalues $D_i^\mu(\tau)$, $N_i^\mu(\tau)$ and $K_i^{j\mu}(\tau)$ ($\mu = 0, 1, 2, 3$), respectively. The quantity τ was time in the comoving frame. Recall that t represents time as measured in the laboratory frame. In this covariant model we have taken

(6*a*) $D_i^\mu(\tau) = D_i(\tau)$, if $\mu = 0$

(6*b*) $= 0$, otherwise

with similar relations for $N_i^{\mu}(\tau)$ and $K_i^{j\mu}(\tau)$. In general, however, the eigenvalues corresponding to $\mu = 1, 2, 3$ may be different from zero (Kamal *et al.*, 1992). The numbers, $D_i(\tau)$, $N_i^{\mu}(\tau)$ and $K_i^{j\mu}(\tau)$ may be considered to be free parameters analogous to damping coefficients, natural frequencies and coupling constants. Φ_i was a 4×1 column vector with the first entry as the only nonzero entry representing the electrical potential φ. Eq. (5) was transformed in the laboratory frame under a similarity transformation (Lorentz transformation) with λ_i as the transformation matrix. Under this similarity transformation the matrices $\boldsymbol{D}_i(\tau)$, $\boldsymbol{N}_i(\tau)$, $\boldsymbol{K}_i^j(\tau)$ and the four-dimensional-spacetime-vector field Φ_i assumed the following form

(7*a*) $\Phi_i \;\Rightarrow\; A_i = \lambda_i\, \Phi_i$

(7*b*) $\boldsymbol{D}_i(\tau) \;\Rightarrow\; \Delta_i(\tau) = \lambda_i\, \boldsymbol{D}_i(\tau)\, \lambda_i^{\sim}$

(7*c*) $\boldsymbol{N}_i(\tau) \;\Rightarrow\; \eta_i(\tau) = \lambda_i\, \boldsymbol{N}_i(\tau)\, \lambda_i^{\sim}$

(7*d*) $\boldsymbol{K}_i^j(\tau) \;\Rightarrow\; \kappa_i^j(\tau) = \lambda_i\, \boldsymbol{K}_i^j(\tau)\, \lambda_i^{\sim}$

where $\lambda^{\sim} = g\lambda g$, $g_{\mu\nu}$ is the metric tensor (μ, $\nu = 0, 1, 2, 3$) with $g_{rr} = 1$; $r = 1, 2, 3$, $g_{00} = -1$, $g_{\mu\nu} = 0$, otherwise. Later on, Eq. (7*e*), which appears after Eq. (13), shall be added to this set. Therefore, Eq. (5) may be written as

(8) $$A_i \frac{d^2 A_i}{d\tau^2} + \Delta_i(\tau)\frac{dA_i}{d\tau} + \eta_i^2(\tau) A_i = \sum{}_j \kappa_i^j(\tau) A_j$$

A_j's are in fact $A_j^{\mu} = (\varphi, \boldsymbol{A})$; $\mu = 0, 1, 2, 3$ – the components of four-dimensional-spacetime-potential-vector field (the components of A_j contain both the electric potential and the magnetic vector potential). The state transition matrix was constructed by defining new variables $Z_k = f(A_k, dA_k/d\tau)$, $k = 1, ..., m$ where $m = 2n$ (n was the number of dendritic trees considered in this model, usually $\sim 10^{15}$). Let us define a dimensionless parameter $t_{scale} = \tau/\Omega$ (Ω is a scaling parameter which may be taken as the average time of travel of a signal between two neurons). The coördinates are, therefore, defined as

(9*a*) $Z_k = A_k$, if k is an odd number

(9*b*) $Z_k = dA_{k-1}/dt_{scale}$, otherwise

In terms of Z_k, the system of equations (8) may be written as

(10) $$\frac{d\boldsymbol{Z}}{dt} = \boldsymbol{A}_{COV}\boldsymbol{Z}$$

where $\boldsymbol{Z} = [Z_k]$ is a column vector and $\boldsymbol{A}_{COV}$ is the state transition matrix. The state transition matrix is a function of D's, N's and K's and Ω. The scaling parameter Ω is introduced to make all the elements of the state transition matrix dimensionless and new variables are defined as $\mathsf{N} = \Omega\eta$, $\mathsf{D} = \Omega\Delta$, $\mathsf{K} = \Omega^2\kappa$. The state transition matrix is a linear transformation. Different matrices could be generated by assigning different values to *D*'s, *N*'s and *K*'s. Each entry in this state transition matrix is itself a 4×4 matrix.

$$
(11)\quad \boldsymbol{A}_{\mathrm{COV}} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\
-N_1^{\,2} & -D_1 & K_1^{\,2} & 0 & K_1^{\,3} & \cdots\cdots & K_1^{\,m} & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots\cdots & 0 & 0 \\
K_2^{\,1} & 0 & -N_2^{\,2} & -D_2 & K_2^{\,3} & \cdots\cdots & K_2^{\,m} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\
K_3^{\,1} & 0 & K_3^{\,2} & 0 & -N_3^{\,2} & \cdots\cdots & K_3^{\,m} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
K_n^{\,1} & 0 & K_n^{\,2} & 0 & K_n^{\,3} & \cdots\cdots & -N_n^{\,2} & -D_n
\end{bmatrix}
$$

Need for Generalized Coupling

In the linear model of Wright & Kydd (1984) the electrical potentials φ_i's are coupled to φ_j's through coupling parameters $K_i^j(t)$. However, we note that a change in potential $d\varphi_j/d\tau$ induces a magnetic field because of flow of current. A magnetic field shall, in turn, exert Lorentz force on a charged particle and hence, in general, $d\varphi_j/d\tau$, shall influence φ_i.

In our covariant model (Kamal, Siddiqui & Husain, 1989) the dependence of A_i 's on A_j's was also suggested when we tried to multiply two state transition matrices (Kamal, 1989). Upon multiplication of two state transition matrices we came up with nonzero coefficients for A_j's.

We are generalizing Wright & Kydd's damped coupled harmonic oscillator equations to include a generalized coupling, which also depends on $d\varphi_j/d\tau$'s. A covariant model is constructed using this coupling and state transition matrix is obtained for the generalized model.

Our Generalized Coupling Model

To introduce the generalized coupling let us modify Eq. (1) as

$$
(12)\qquad \frac{d^2\varphi_i}{dt_2} + D_i(t)\frac{d\varphi_i}{dt} + N_i^2(t)\varphi_i = \Sigma_j\left[K_i^i(t)\varphi_j + M_i^j(t)\frac{d\varphi_j}{dt}\right]
$$

The $M_i^j(t)$'s are free parameters given physiological meaning under the assumptions that they have a finite variance σ_M about $\overline{M}$ and they are stochastically independent. In the next section we are going to set up a generalized coupling model using the covariant formulation developed earlier.

COVARIANT GENERALIZED COUPLING MODEL

In the spirit of four-dimensional vector field formulation used in relativity it is desirable to express the electrodynamical equations representing global electrocortical activity in covariant form. *Covariant Generalized Coupling Model* is devised to serve this purpose.

Mathematical Formulation

To set up a covariant formulation let us write the electrical potential variation for a mass of unit sources coupled to each other in the comoving frame of signal passing through a segment of the dendritic tree as [compare with Eq. (5)]

$$\text{(13)} \qquad \frac{d^2\Phi_{\rm i}}{{\rm d}\tau^2} + \boldsymbol{D}_{\rm i}(\tau)\frac{d\Phi_{\rm i}}{{\rm d}\tau} + \boldsymbol{N}_{\rm i}^2(\tau)\Phi_{\rm i} = \Sigma_{\rm j}\left[\boldsymbol{K}_{\rm i}^{\rm j}(\tau)\Phi_{\rm j} + \boldsymbol{M}_{\rm i}^{\rm j}(\tau)\frac{d\Phi_{\rm j}}{{\rm d}\tau}\right]$$

where $\boldsymbol{M}_{\rm i}^{\rm j}(\tau)$'s are 4×4 matrices having eigenvalues $M_{\rm i}^{\rm j\mu}(\tau)$; $\mu = 0, 1, 2, 3$. No particular type of distribution for $M_{\rm i}^{\rm j\mu}(\tau)$ is assumed. All other symbols are defined after Eq. (5).

A similarity transformation under $\lambda_{\rm i}$ transforms the various spacetime vector fields and matrices as given in Eq. (7*a* - *d*). The $\boldsymbol{M}_{\rm i}^{\rm j}(\tau)$'s transform as

$$\text{(7}e\text{)} \qquad \boldsymbol{M}_{\rm i}^{\rm j}(\tau) \Rightarrow \mu_{\rm i}^{\rm j}(\tau) = \lambda_{\rm i}\, \boldsymbol{M}_{\rm i}^{\rm j}(\tau)\, \lambda_{\rm i}^{\sim}$$

Eq. (13), therefore, becomes [compare with Eq. (8)]

$$\text{(14)} \qquad \frac{d^2 A_{\rm i}}{{\rm d}\tau^2} + \Delta_{\rm i}(\tau)\frac{dA_{\rm i}}{{\rm d}\tau} + \eta_{\rm i}^2(\tau)A_{\rm i} = \Sigma_{\rm j}\left[\kappa_{\rm i}^{\rm j}(\tau)A_{\rm j} + \mu_{\rm i}^{\rm j}(\tau)\frac{dA_{\rm i}}{{\rm d}\tau}\right]$$

Introducing the generalized coördinates defined in Eq. (9*a*, *b*) we obtain an eigenvalue equation of the form of Eq. (10). The state transition matrix is now a function of *D*'s, *N*'s, *K*'s, *M*'s and Ω. If we introduce $\mathrm{M}_{\rm i}^{\rm j}(\tau) = \Omega\, \mu_{\rm i}^{\rm j}(\tau)$ to make all the elements dimensionless, the generalized coupling state transition matrix, $\mathbb{A}_{\rm CGC}$, becomes

$$\text{(15)} \quad \mathbb{A}_{\rm CGC} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\mathrm{N}_1^2 & -\mathrm{D}_1 & \mathrm{K}_1^2 & \mathrm{M}_1^2 & \mathrm{K}_1^3 & \cdots & \mathrm{K}_1^m & \mathrm{M}_1^m \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \mathrm{K}_2^1 & \mathrm{M}_2^1 & -\mathrm{N}_2^2 & -\mathrm{D}_2 & \mathrm{K}_2^3 & \cdots & \mathrm{K}_2^m & \mathrm{M}_2^m \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \mathrm{K}_3^1 & \mathrm{M}_3^1 & \mathrm{K}_3^2 & \mathrm{M}_3^2 & -\mathrm{N}_3^2 & \cdots & \mathrm{K}_3^m & \mathrm{M}_3^m \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\llcorner \quad K_n^1 \quad M_n^1 \quad K_n^2 \quad M_n^2 \quad K_n^3 \quad \ldots\ldots\ldots -N_n^2 \quad -D_n \quad \lrcorner$$

Compare Eq. (15) with Eq. (11). The state transition matrix is a linear transformation and the set of state transition matrices forms a group under the operation of matrix multiplication. The identity of this group corresponds to the physiological state of *brain death*. This can be shown easily for this set of state transition matrices by applying a reasoning similar to one presented earlier for the covariant model (Siddiqui *et al.*, 1993).

Predictions —— *Effects of Weak External Magnetic Field*

Consider the effects of a weak magnetic field $\boldsymbol{B}_{ext}$ ($\sim 10^{-16}$ tesla) which is assumed to be uniform throughout the region concerned (The ambient magnetic field noise is much greater than 10^{-16} tesla. Therefore, a magnetically shielded room is needed to perform any measurement related to these weak fields). The magnetic field, not varying with time, is generated by a vector potential such that $\boldsymbol{B}_{ext} = \nabla \times \boldsymbol{A}_{ext}$. (In order to calculate the shifts in frequencies based on a certain form of magnetic vector potential, a magnetic field corresponding to that potential can always be determined using the above equation)**. The four-dimensional spacetime vector potential field is $\boldsymbol{A}_{ext}$. In the presence of this field the natural frequencies $N_i(\tau)$ will be modified to $N'_i(\tau)$. However, the damping coefficients and couplings of the individual A's and $(dA/d\tau)$'s will remain unchanged. Klitzing (1989) has shown that the effects of weak magnetic fields on the EEG of man are large enough to be measured. Therefore, Eq. (14) takes the following form in the laboratory frame

(16) $$\frac{d^2A'_i}{d\tau^2} + \Delta_i(\tau)\frac{dA'_i}{d\tau} + \eta_i'^2(\tau)A'_i = \sum{}_j \left[\kappa_i^j(\tau)A'_j + \mu_i^j(\tau)\frac{dA'_i}{d\tau}\right]$$

where $A'_i = A_i + A_{ext}$. Since the signal velocities are very small as compared to the velocity of light, we take $t \approx \tau$. The external field $A_{ext}(x, y, z)$ does not depend on time. We, therefore, have

$$\frac{d^2A'_i}{d\tau^2} = \frac{d^2A_i}{d\tau^2}, \frac{dA'_i}{d\tau} = \frac{dA_i}{d\tau}$$

The matrices $\delta\eta_i^{2}$'s are 4×4 matrices having eigenvalues δ_i^{μ}; $\mu = 0, 1, 2, 3$, representing the shifts in frequencies. Subtracting Eq. (14) from Eq. (16) and introducing, $\eta'_i{}^2 = \eta_i^2 + \delta\eta_i^2$, we have

** Although the EEG sources are not equivalent to isolated oscillating charges in a dielectric medium, an equivalent 4-potential can be found out for the electric fields in the brain. In fact, EEG gives the average effect and hence the effects of interaction of an external magnetic field with this equivalent four-dimensional spacetime vector potential field could be computed.

(17) $$\delta\eta_i^2 A'_i = -(\eta_i^2 - \sum_j \kappa_i^j) A_{ext}$$

We assume that $\delta\eta_i$, κ_i^j and η_i can all be simultaneously diagonalized. In other words, $[\delta\eta_i, \kappa_i^j] = 0$, $[\delta\eta_i, \eta_i] = 0$ etc. Applying the transformation $\delta\eta_i \Rightarrow \lambda_i^{\sim}\, \delta\eta_i\, \lambda_i = \delta\boldsymbol{N}_i$ *etc*. to write Eq. (17) in the comoving frame of the signal, we

(18) $$\delta\boldsymbol{N}_i^2 A_i'^{\sim} = -(\boldsymbol{N}_i^2 - \sum_j \boldsymbol{K}_i^j)(A_{ext})_i^{\sim}$$

where $A'^{\sim}_i = A_i + (A_{ext})_i^{\sim}$; $(A_{ext})_i^{\sim} = \lambda_i^{\sim} A_{ext}$. The matrices $\delta\boldsymbol{N}_i$, $\boldsymbol{N}_i$ and $\boldsymbol{K}_i^j$ are already diagonalized. Eq. (18) shall yield four equations each one sufficient to determine an eigenvalues δ_i^μ. The results are

(19) $$(\delta_i^\mu)^2 = [\Sigma_j K_i^{j\mu} - (N_i^\mu)^2]\ \Theta_i^\mu$$

where $\Theta_i^0 = -\gamma_i \boldsymbol{v}_i \cdot \boldsymbol{A}_{ext}/(\varphi_i - \gamma_i \boldsymbol{v}_i \cdot \boldsymbol{A}_{ext})$, $\Theta_i^r = 1$; $r = 1, 2, 3$.

This is the same result we obtained in the covariant model (Kamal, Siddiqui & Husain, 1992). We note that all the factors related to generalized coupling with $dA/d\tau$ have canceled out. Therefore, we conclude that a weak, uniform, stationary magnetic field shall give the same first-order shift in frequencies in the presence of generalized coupling.

Suggested Test for Covariant Generalized Coupling Model

In order to see the effects of generalized coupling on the frequency shift we consider a magnetic field $\boldsymbol{B}_{ext}(t)$ which is generated by a vector potential $\boldsymbol{A}_{ext}(t)$ varying linearly with time. Defining a four-dimensional spacetime vector potential field A_{ext}, we may write Eq. (5) in the presence of a weak magnetic field. Since $A'_i = A_i + A_{ext}$, and $A_{ext}(x, y, z, t)$ has a linear time dependence, we have

$$\frac{d^2 A'_i}{dt^2} = \frac{d^2 A_i}{dt^2}, \frac{dA'_i}{dt} \neq \frac{dA_i}{dt}$$

Subtracting Eq. (14) from Eq. (16) and introducing $\delta\eta_i$ as the first-order correction to η_i, we have [compare with Eq. (17)]

(20) $$\delta\eta_i^2 A'_i = -(\eta_i^2 - \sum_j \kappa_i^j) A_{ext} - (\Delta_i - \sum_j \mu_i^j) \frac{dA_{ext}}{dt}$$

If all the matrices can be simultaneously diagonalized, Eq. (20) may be written in the comoving frame of the signal as

(21) $$\delta\boldsymbol{N}_i^2 A_i'^{\sim} = -(\boldsymbol{N}_i^2 - \sum_j \boldsymbol{K}_i^j)(A_{ext})_i^{\sim} - (\boldsymbol{D}_i - \sum_j \boldsymbol{M}_i^j)\left(\frac{dA_{ext}}{dt}\right)_i^{\sim}$$

where $(dA_{ext}/dt)_i^{\sim} = \lambda_i^{\sim}\, dA_{ext}/dt$. The matrices $\delta\boldsymbol{N}_i$, $\boldsymbol{N}_i$, $\boldsymbol{K}_i^j$ and $\boldsymbol{M}_i^j$ are already diagonalized. Eq. (21) shall yield four equations, which may be solved to determine the eigenvalues δ_i^μ. The results are

(22) $(\delta_i^{\mu})^2 = [\Sigma_j K_i^{j\mu} - (N_i^{\mu})^2]\, \Theta_i^{\mu} + [\Sigma_j M_i^{j\mu} - (D_i^{\mu})^2]\, \zeta_i^{\mu}$

The factors ζ_i^{μ} are given by

(23*a*) $\zeta_i^0 = (\gamma_i\, d\varphi_{ext}/dt - \gamma_i\, \boldsymbol{v}_i \cdot d\boldsymbol{A}_{ext}/dt)/(\gamma_i\, \varphi_{ext} + \varphi_i - \gamma_i\, \boldsymbol{v}_i \cdot \boldsymbol{A}_{ext})$

(23*b*-*d*) $\zeta_i^r = [\Im(A^r_{ext})]^{-1}\, d\Im(A^r_{ext})/dt;\ r = 1, 2, 3$

with $\Im(A^r_{ext}) = -\gamma_i\, \varphi_{ext} + A^r_{ext} + (\boldsymbol{v}_i \cdot \boldsymbol{A}_{ext})\, v^r\, (\gamma_i - 1)/|\boldsymbol{v}_i|^2$.

Before discussing the set of equations (22) let us first consider what do we mean by shifts in frequencies of EEG when we know that there are no well-defined frequencies in EEG. Although there is not a single well defined frequency in EEG every rhythm has a finite range and variance of frequencies. Therefore, according to Cramer's Central Limit Theorem (Kamal, 1989; Wright & Kydd, 1984) the frequencies may be assumed to be clustered around their average values. For the purpose of testing this model shifts in the frequencies may be interpreted as shifts in the average frequencies. The shifts in frequencies, given by Eq. (22), may represent measure of strength of coupling through the factor,$[\Sigma\, K_i^{j\mu} - (N_i^{\mu})^2]$

a. If the coupling is strong i.e. $\Sigma\, K_i^{j\mu} > (N_i^{\mu})^2$, a slight increase in frequencies may be expected.

b. If $\Sigma\, K_i^{j\mu} = (N_i^{\mu})^2$, frequencies may not be shifted.

c. If $\Sigma\, K_i^{j\mu} < (N_i^{\mu})^2$, a decaying exponential may be found in the EEG spectrum.

Therefore, one may observe modulation of frequencies. Note that field is applied in the *x* direction. If its magnitude is changed from $-\varepsilon$ to $+\varepsilon$ in a period of time Δt, *i. e.*, on the average $|\boldsymbol{A}_{ext}| = 0$, we expect that

$$\delta_i^0 \propto \sqrt{\left|\frac{d\boldsymbol{A}_{ext}}{dt}\right|}$$

Using Cramer's Central Limit Theorem to replace the individual frequencies by their averages, we get

(24) $$\overline{\delta^0} \propto \sqrt{\left|\frac{d\boldsymbol{A}_{ext}}{dt}\right|}$$

This prediction may be checked experimentally. An experiment may be designed in which a magnetic field varying linearly with time

(25) $$\boldsymbol{A}_{ext} = \boldsymbol{A}_0\, (a + bt)$$

may be applied. For 100 different sets of *a* and *b* the shifts in frequencies $< \delta^0 >$ may be noted. Correlation of $\overline{\delta^0}$ and $\sqrt{\left|\frac{d\boldsymbol{A}_{ext}}{dt}\right|} = \sqrt{|\boldsymbol{A}_0 b|}$ may, then, be studied. Klitzing (1989) suggests that static magnetic fields increase the power intensity of EEG of man. However, no studies have been conducted to study the shifts as predicted in this model.

Because of the presence of the factor $(A_{ext})^{-1}\, dA_{ext}/dt$ in Eq. (23*b*-*d*), the frequencies will become unbounded when $|\boldsymbol{A}_{ext}| = 0$. This is not possible, physically. We, therefore, conclude that

$$\Sigma_j M_i^{jr} = D_i^r;\ r = 1, 2, 3 \tag{26}$$

This may be considered as a condition of *impedance matching*.

DISCUSSION AND CONCLUSION

Generalized coupling model presented here proposes a new mechanism for the interactions of neurons responsible for the output of EEG. Since EEG provides statistical information we have to obtain a statistical estimate to be able to handle such a large number of parameters. The number of parameters in the model may, however, be considerably reduced by considering the nearest neighbor interaction. Wright and Kydd (1984) have considered $n = 50$ to calculate the power spectrum in their linear model. The same procedure may be applied to this model.

At present we are trying to simulate coupled harmonic oscillators to be compared later with the digitized electroencephalograms provided by local neurologists (Siddiqui, Kamal & Khan, 1990). This endeavor may give us a better idea of how the estimates of *D*i's, *N*i's, *K*ij's and *M*ij's would be constrained by the biological data. Another approach may be to study system response by transfer function approach. Transfer functions of the various models considered in this paper are given in the appendix.

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APPENDIX: Transfer Functions

In the following listing first equation represents the basic equation representing coupled oscillations. Second equation gives the transfer function.

Linear Model of Wright and Kydd

$$\frac{d^2\varphi_i}{dt_2} + D_i(t)\frac{d\varphi_i}{dt} + N_i^2(t)\varphi_i = \sum{}_j K_i^i(t)\varphi_j,\ \frac{\varphi_i}{\varphi_j} = \frac{K_i^i(t)}{\boldsymbol{D}^2 + D_i(t)\boldsymbol{D} + N_i^2(t)},\ \text{where } \boldsymbol{D}^n = \frac{d^n}{dt^n}$$

Generalized Coupling Model

$$\frac{d^2\varphi_i}{dt_2}+D_i(t)\frac{d\varphi_i}{dt}+N_i^2(t)\varphi_i=\sum{}_j\left[K_i^i(t)\varphi_j+M_i^j(t)\frac{d\varphi_j}{dt}\right],\ \frac{\varphi_i}{\varphi_j}=\frac{K_i^i(t)+M_i^j(t)\boldsymbol{D}}{\boldsymbol{D}^2+D_i(t)\boldsymbol{D}+N_i^2(t)}$$

Covariant Model

$$A_i\frac{d^2A_i}{d\tau^2}+\Delta_i(\tau)\frac{dA_i}{d\tau}+\eta_i^2(\tau)A_i=\sum{}_j\kappa_i^j(\tau)A_j,\ \frac{A_i^\mu}{A_j^\nu}=\frac{\sum_\beta[K_i^i(\tau)]^\beta}{\boldsymbol{D}^2+\sum_\alpha\{[\Delta_i(\tau)]_\mu^\alpha\boldsymbol{D}+[\eta_i^2(\tau)]_\mu^\alpha\}}$$

where α, β run from 0 to 3 and $\boldsymbol{D}^n=\dfrac{d^n}{dt^n}$

Covariant Generalized Coupling Model

$$\frac{d^2A_i}{d\tau^2}+\Delta_i(\tau)\frac{dA_i}{d\tau}+\eta_i^2(\tau)A_i=\sum{}_j\left[\kappa_i^j(\tau)A_j+\mu_i^j(\tau)\frac{dA_i}{d\tau}\right],\ \frac{A_i^\mu}{A_j^\nu}=\frac{\sum_\beta\{[K_i^i(\tau)]^\beta+[\mu_i^j(\tau)]_\nu^\beta\boldsymbol{D}\}}{\boldsymbol{D}^2+\sum_\alpha\{[\Delta_i(\tau)]_\mu^\alpha\boldsymbol{D}+[\eta_i^2(\tau)]_\mu^\alpha\}}$$

REFERENCES

Elul, R. (1972). The genesis of EEG. *Int. Rev. Neurobiol.* **15**: 227-272

Kamal, S. A. (1989). Spacetime representation in the brain. *Ph.D. dissertation*, 1989, Univ. Karachi (unpublished), *full text***:** http://eprints.hec.gov.pk/1074/1/803.html.htm

Kamal, S. A., Siddiqui, K. A. & Husain, S. A. (1989). Space-time representation of global electrocortical activity. *Biol. Cybern.* **60**: 307-309, *abstract***:** http://www.ngds-ku.org/pub/jourabst1.htm#J08: *full text***:** http://www.ngds-ku.org/Papers/J08.pdf

Kamal, S. A., Siddiqui, K. A. & Husain, S. A. (1992). Effects of weak magnetic fields on global electrocortical activity. *Kar. U. J. Sc.* **20**: 31-35, *abstract***:** http://www.ngds-ku.org/pub/jourabst0.htm#J12: *full text***:** http://www.ngds-ku.org/Papers/J12.pdf

Klitzing, L. (1989). Static magnetic fields increase the power intensity of EEG of man. *Brain Res.* **483**: 201-203

Siddiqui, K. A., Kamal, S. A., Husain, S. A. & Khan, N. U. (1993). Group structure of a covariant model of global electrocortical activity. *Kar. U. J. Sc.* **21**: 35-38, *abstract***:** http://www.ngds-ku.org/pub/jourabst0.htm#J14: *full text***:** http://www.ngds-ku.org/Papers/J14.pdf

Siddiqui, K. A., Kamal, S. A. & Khan, N. U. (1990). Neurophysics: A beginner's viewpoint. *Proc. Second National Symposium on Frontiers in Physics*, ed. by G. A. Murtaza & M. A. Baig, Quaid-é-Azam Univ., Islamabad (Pakistan), pp. 285-305, *abstract***:** http://www.ngds-ku.org/pub/confabst0.htm#C35: *full text***:** http://www.ngds-ku.org/Papers/C35.pdf

Wright, J. J. & Kydd, R. R. (1984). A linear theory of global electrocortical activity and its control by the lateral hypothalamus. *Biol. Cybern.* **50**: 75-82

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