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Spacetime Representation of Global Electrocortical Activity

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Abstract. A model for global electrocortical activity is developed by considering telencephalonic structures as mass of linked oscillators generating activity with a number of resonant modes. Equations for the signals are written in the comoving frame and then transformed into the laboratory frame. The state transition matrix is obtained in the presence of electric and magnetic fields.

1 Introduction

Wright and Kydd (1984) developed a linear model for global electrocortical activity and its control by lateral hypothalamus. Their model rests on drastic simplifications. It bypasses issues of cell-to-cell coupling, details of anatomy etc. Moreover it does not take into account of the magnetic fields which are generated when there is a motion of charges.

We start with the model of Wright and Kydd (1984), but write the equation for the potential of a segment of dendritic tree in the comoving frame of the signal. When this equation is transformed into the laboratory frame a magnetic vector potential appears along with electrostatic potential. Because of the appearance of this magnetic vector potential the eigenvalues are modified.

2 Essential Theoretical Features

The essential theoretical features of this model, somewhat modified from Wright et al. (1984), can be summarized as:

(a) Electrocortical recordings reflect the transformed spatial average of cortical potentials (Elul 1972).

(b) The fixed circuitry of the telencephalon and stochastic considerations of the linkages between neural elements render the telencephalon a linear wave medium, with regard to the gross wave potentials, although the underlying microscopic interactions may be extremely non-linear.

(c) Closed and constant boundary conditions lead the linear waves to generate activity at a large number of resonant modes, each associated with a constant natural frequency (and presumably a specific spatial configuration). These are independent properties for time-invariant linear systems (Kuo 1982; Feynman et al. 1963).

(d) Consideration of the modular and architectonic orderliness of the telencephalon requires that the values for the natural frequencies of the resonant modes be clustered about certain central values, each cluster within the frequency band width of a major cerebral rhythm.

(e) Ascending inhibitory systems act partly to damp resonant activity and partly as a source of noise-like driving signals, by their input to telencephalon at their fields of termination.

(f) An electrical potential in a comoving frame of the signal transforms as four-potential in the laboratory frame.

3 Mathematical Description

3.1 Covariant Description of Potentials

Let us introduce 4-potential A^{μ}; $\mu = 0, 1, 2, 3$ which can be written as

$$A^{\mu} = \begin{bmatrix} \Phi \\ A \end{bmatrix}. \tag{1}$$

In the subsequent discussion the superscript μ is dropped. In the comoving frame of the signal passing through a segment of the dendritic tree, the electrical potential can be represented as

$$\ddot{\phi}_i + D_i(\tau)\dot{\phi}_i + N_i^2(\tau)\phi_i = 0. \tag{2}$$

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The Roman index i which runs from 1 to n denotes the number of dendritic trees considered in the model. τ is time in the comoving frame of the signal. There is no summation over repeated Roman indices. However, repeated Greek indices denote summation. The parameters $D_i(\tau)$ and $N_i(\tau)$ are free parameters analogous to a damping coefficient and a natural frequency. In matrix form the above equation can be written as

$$\ddot{\Phi}_i + \mathscr{D}_i(\tau)\dot{\Phi}_i + \mathscr{N}_i^2(\tau)\Phi_i = 0, \tag{3}$$

where $\Phi_i = \begin{bmatrix} \phi_i \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\dot{\Phi}_i = d\Phi_i/d\tau$, $\ddot{\Phi}_i = d^2\Phi_i/d\tau^2$ etc. and

$$\mathscr{D}_i(\tau) = \begin{bmatrix} D_i(\tau) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4a}$$

$$\mathscr{N}_i(\tau) = \begin{bmatrix} N_i(\tau) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4b}$$

Now we transform (3) in the laboratory frame. Under the Lorentz transformation λ_i the four vectors and matrices transform as

$$\Phi_i \to A_i = \lambda_i \Phi_i, \tag{5a}$$

$$\mathscr{D}_i(\tau) \to \Delta_i(\tau) = \lambda_i \mathscr{D}_i(\tau)\lambda_i^T, \tag{5b}$$

$$\mathscr{N}_i(\tau) \to \eta_i(\tau) = \lambda_i \mathscr{N}_i(\tau)\lambda_i^T, \tag{5c}$$

where λ^T denotes the transpose of matrix λ. Therefore Eq. (3) transforms in the laboratory frame as

$$\ddot{A}_i + \Delta_i(\tau)\dot{A}_i + \eta_i^2(\tau)A_i = 0. \tag{6}$$

A_i, $\dot{A}_i$, and $\ddot{A}_i$ are all four vectors, $\Delta_i(\tau)$ and $\eta_i(\tau)$ are 4×4 matrices.

A mass of unit sources coupled to each other may be similarly represented by (j runs from l to n subject to the condition that $i \neq j$).

$$\ddot{A}_i + \Delta_i(\tau)\dot{A}_i + \eta_i^2(\tau)A_i = \sum_j K_j^i(\tau)A_j, \tag{7}$$

where $\Delta_i(\tau)$, $\eta_i(\tau)$, $K_j^i(\tau)$ are free parameters (4×4 matrices).

3.2 Development of State Transition Matrix

Let $\Omega_1 = A_1$, $\Omega_2 = \dot{\Omega}_1 = \dot{A}_1$, $\Omega_3 = A_2$, $\Omega_4 = \dot{\Omega}_3 = \dot{A}_2$ to $\Omega_{m-1} = A_n$, $\Omega_m = \dot{\Omega}_{m-1} = \dot{A}_n$; $m = 2n$. In matrix representation

$$dZ/d\tau = AZ, \tag{8}$$

where

$$Z = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \\ \Omega_5 \\ \Omega_6 \\ \vdots \\ \Omega_m \end{bmatrix}; \tag{9a}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \ldots 0 & 0 \\ -\eta_1^2 & -\Delta_1 & K_2^1 & 0 \ldots K_m^1 & 0 \\ 0 & 0 & 0 & 1 \ldots 0 & 0 \\ K_1^2 & 0 & -\eta_2^2 & -\Delta_2 \ldots K_n^2 & 0 \\ 0 & 0 & 0 & 0 \ldots 0 & 0 \\ K_1^3 & 0 & K_2^3 & 0 \ldots K_m^3 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \ldots 0 & 1 \\ K_1^n & 0 & K_2^n & 0 \ldots -\eta_n^2 & -\Delta_n \end{bmatrix}. \tag{9b}$$

The matrix A is a $4m \times 4m$ matrix and it will have $4m$ eigenvalues. If α represents an eigenvalue of the matrix A, the secular equation can be written as

$$|\alpha I - A| = 0, \tag{10}$$

where I is a $4m \times 4m$ unit matrix. In the subsequent discussion we are dropping the subscript i which denotes the number of dendritic trees considered in the model. Let us take λ in the form

$$\lambda_{00} = \gamma, \quad \lambda_{0p} = \lambda_{p0} = \gamma v_p \qquad (p, q = 1, 2, 3), \tag{11a, b}$$

$$\lambda_{pq} = \delta_{pq} + v_p v_q(\gamma - 1)/v^2 \qquad (c = 1), \tag{11c}$$

where δ_{pq} is kronecker delta given by

$$\begin{aligned} \delta_{pq} &= 1 \quad \text{if} \quad p = q \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{12}$$

and

$$\gamma = (1 - v^2)^{-1/2}. \tag{13}$$

The form of λ chosen in (11a–c) gives

$$\Delta = \gamma^2 D \exists, \tag{14}$$

where $\exists$ is a 4×4 matrix whose elements are given by $\exists_{00} = 1$,

$\exists_{p0} = \exists_{0p} = v_p$, $\exists_{pq} = v_p v_q$.

Taking a boost in the z-direction and calculating the eigenvalues gives the following relationship between the laboratory frame and comoving frame damping coefficients

$$D_{\text{lab}} = D_{\text{comoving}}(1+v^2)/(1-v^2). \quad (15)$$

A similar calculation gives

$$N_{\text{lab}} = N_{\text{comoving}}(1+v^2)/(1-v^2). \quad (16)$$

Note that in the limit $v \to 0$, $D_{\text{lab}} \to D_{\text{comoving}}$ and $N_{\text{lab}} \to N_{\text{comoving}}$.

3.3 Some Special Cases

In this section we discuss the cases when $v \to 0$, $v \to 1$ and finally $v \ll 1$.

Case 1: $v \to 0$. In this case $D_{\text{lab}} \to D_{\text{comoving}}$, $N_{\text{lab}} \to N_{\text{comoving}}$ and our results are identical with the results of Wright and Kydd (1984).

Case 2: $v \to 1$. $D_{\text{lab}} \to \infty$, $N_{\text{lab}} \to \infty$. The left and right average power spectrum obtained before and after unilateral hypothalamic lesion in as close as possible to steady-state conditions can be written as (Wright and Kydd 1984)

$$G^2(\omega) = v_{LA}^2/v_{LB}^2 : v_{CA}^2/v_{CB}^2(\omega). \quad (17)$$

ω is the frequency. The subscripts LA, LB, CA, CB indicate the lesion and control sides, after and before lesion. Substituting the theoretical expression of v's and taking the limit $v \to 1$, we have

$$G^2(\omega) = 1. \quad (18)$$

This shows that there is no effect of lesion on EEG recordings. However, in reality this situation is rare because the velocities of signals are very much less than the velocity of light (Pellionisz and Llinas 1982).

Case 3: $v \ll 1$. In this case we can obtain first order approximation for the expression of the damping coefficients and the natural frequencies.

$$D_{\text{lab}} \approx (1+2v^2)D_{\text{comoving}};$$
$$N_{\text{lab}} = (1+2v^2)N_{\text{comoving}}. \quad (19)$$

4 Discussion and Conclusion

The model presented here is a modification of Wright and Kydd's linear model. The electrical potentials in the comoving frames transform as four potential in the lab frame introducing magnetic fields along with electrical fields. The state transition matrix and its eigenvalues are modified. A correspondence is established between our results and the results of Wright and Kydd (1984). It is shown that our results reduce to the results of Wright and Kydd when the signal velocity is vanishingly small. The case $v \to 1$ is not of much interest physically. However, the first order corrections to Wright and Kydd's results may be studied to see if they give a better fit to the data. Once we have four potential, different gauges *(Coulomb or Lorentz)* could be used to simplify the resulting wave equations. Further work in this area may include development of a wave equation, quantization of the results presented here.

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