THE MATRIX AND TENSOR QUARTERLY

A publication devoted to the application of

DETERMINANTS · MATRICES · VECTORS · DYADICS & TENSORS

to the Engineering and Physical Sciences

Issued for private circulation to Members by

THE TENSOR SOCIETY OF GREAT BRITAIN

Volume 31 **MARCH 1981** **Number 3**

CONTENTS

DETERMINANT OF GENERAL TENSOR

by

S. A. Kamal *

A generalized definition of a determinant of mth rank is proposed.

While working with tensors, it is sometimes necessary to introduce determinants of tensors of order higher than two. Let us see how the determinant of a second rank tensor (matrix) looks like

$$\det(T^{ab}) = \epsilon_{i_1 i_2 \dots i_n} T^{1 i_1} T^{2 i_2} \dots T^{n i_n} \tag{1}$$

where $a, b = 1, 2, \dots, n$ and $T^{1 i_1}$ are the components of T in the first row and i_1-th column. $\epsilon_{i_1 i_2 \dots i_n}$ is the Levi-Civita tensor, which is of nth rank, and completely antisymmetric. Since

$$\epsilon_{1 2 \dots n} = +1 \tag{2}$$

we can write (1) as

$$\det(T^{ab}) = \epsilon_{1 2 \dots n} \epsilon_{i_1 i_2 \dots i_n} T^{1 i_1} T^{2 i_2} \dots T^{n i_n} \tag{3}$$

This suggests that we can define the determinant of the mth rank tensor as the product of two terms

$$\det(T^{a_1 a_2 \dots a_m}) = (\text{product } \epsilon)(\text{product } T) \tag{4}$$

where

$$(\text{product } \epsilon) = (\epsilon_{1 2 \dots n})(\epsilon_{i_1^{(2)} i_2^{(2)} \dots i_n^{(2)}}) \dots (\epsilon_{i_1^{(m)} \dots i_n^{(m)}})$$

$$(\text{product } T) = (T^{1 i_1^{(2)} i_1^{(3)} \dots i_1^{(m)}}) \dots (T^{n i_n^{(2)} i_n^{(3)} \dots i_n^{(m)}})$$

$a_1, a_2, \dots, a_m = 1, 2, \dots, n$ and $i_1^{(p)}$, $p = 1, 2, \dots, m$ are the values that the indices $a_1, a_2, \dots, a_m$ take. We take $i_1^{(1)} = 1$, $i_2^{(1)} = 2$, $\dots$, $i_n^{(1)} = n$. This can also be written in compact form

$$\det(T^{a_1 a_2 \dots a_m}) = \prod_{p=1}^{m} (\epsilon_{i_1^{(p)} i_2^{(p)} \dots i_n^{(p)}}) \prod_{q=1}^{n} (T^{i_q^{(1)} i_q^{(2)} \dots i_q^{(m)}}) \tag{5}$$

Note that instead of rows and columns in a matrix we can obtain different 'arrays' by fixing each of the $a_1, a_2, \dots, a_m$ one-by-one. It can be easily verified that

(a) the determinant is zero if any 'array' consists of zeros;

(b) the determinant is multiplied by a number A if any 'array' is multiplied by A;

(c) the determinant changes sign by the interchange of two adjacent 'arrays';

(d) the determinant vanishes if two 'arrays' are identical.

* Department of Physics, Indiana University, Bloomington, Indiana 47405, USA.

We now prove the following results.

Theorem 1: The determinant of an inner product is equal to the determinant of the outer product.

For a second rank tensor

$$\det(A^{ab}) = \epsilon_{a_1 a_2 \ldots a_n} \epsilon_{b_1 b_2 \ldots b_n} A^{a_1 b_1} A^{a_2 b_2} \ldots A^{a_n b_n} \tag{6}$$

Now

$$\det(A^{ab}) \det(B^{cd}) = (\epsilon_{a_1 \ldots a_n} \epsilon_{b_1 \ldots b_b} \epsilon_{c_1 \ldots c_n} \epsilon_{d_1 \ldots d_n}) \times (A^{a_1 b_1} B^{c_1 d_1} \ldots A^{a_n b_n} B^{c_n d_n})$$

If we replace c by b and sum over b we have

$$\det(A^{ab}) \det(B^{bd}) = [\epsilon_{a_1 \ldots a_n} (\epsilon_{b_1 \ldots b_n})^2 \epsilon_{d_1 \ldots d_n}] \times [A^{a_1 b_1} B^{b_1 d_1} \ldots A^{a_n b_n} B^{b_n d_n}]$$

$(\epsilon_{b_1 \ldots b_n})^2 = +1$ when it is non-vanishing. In those cases we may define

$$A^{a_i b_i} B^{b_i d_i} = (AB)^{a_i d_i} \tag{7}$$

as the inner product of A and B. Therefore

$$\det(A^{ab}) \det(B^{bd}) = \det(AB^{ad}) \tag{8}$$

Note that we can change contravariant tensors to covariant (for their contraction) by multiplication with $\eta_{\mu\nu}$ having a determinant -1. If we define

$$A^{a_i b_i} B^{c_i d_i} = (AB)^{a_i b_i c_i d_i} \tag{9}$$

as the outer product of tensors A and B, we notice that

$$\det(A^{ab}) \det(B^{cd}) = \det(AB^{abcd}) \tag{10}$$

We note that determinant is a number which does not depend on the choice of indices and so from equations (8) and (10) we have

$$\det(A^{ab}) \det(B^{bd}) = \det(A^{ab}) \det(B^{cd}) \tag{11}$$

and so we have

$$\det(AB^{abcd}) = \det(AB^{ab}) \tag{12}$$

Now we prove this theorem for a general tensor of rank n. Suppose it is true for a tensor of rank k,

$$\det(T^{a_1 a_2 \ldots a_k a_{k+1} a_{k+2}}) = \det(T^{a_1 a_2 \ldots a_k}) \tag{13}$$

Now we take a new tensor whose components are given by C^{a_0} which is of first rank. Therefore according to equation (13)

$$\det(C^{a_0}) \det(T^{a_1 a_2 \ldots a_k a_{k+1} a_{k+2}}) = \det(C^{a_0}) \det(T^{a_1 a_2 \ldots a_k})$$

By definition (5) we see that the two sides are the determinants of tensors $(CT)^{a_0 a_1 \cdots a_{k+2}}$ and $(CT)^{a_0 a_1 \cdots a_k}$ and so

$$\det(CT^{a_0 a_1 \cdots a_{k+2}}) = \det(CT^{a_0 a_1 \cdots a_k}) \qquad (14)$$

Therefore the assumption that the theorem is true for k implies that it is true for $k+1$ because on the right-hand side we have a tensor of rank $k+1$. We already proved that it is true when $k = 2$. Hence it is true for all $n > 2$. For $n = 1$ the result can be easily proved.

Theorem 2: The determinants of a general tensor form an abelian group under multiplication.

Let $\det(A^{a_1 a_2 \cdots a_m})$ and $\det(B^{b_1 b_2 \cdots b_n})$ be two elements in set D.

(i) Using theorem 1 we can show that

$$\det(A^{a_1 a_2 \cdots a_m})\det(B^{b_1 b_2 \cdots b_n}) = \det(AB^{a_1 \cdots a_m b_1 \cdots b_n}) \in D$$

(ii) For all $\det(A^{a_j})$, $\det(B^{b_k})$ and $\det(C^{c_l})$ in D, $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$; $l = 1, 2, \ldots, p$. Therefore

$$\det(A^{a_j})[\det(B^{b_k})\det(C^{c_l})] = \det(ABC^{a_j b_k c_l}) = [\det(A^{a_j})\det(B^{b_k})]\det(C^{c_l})$$

by theorem 1.

(iii) There exists 1 in D such that

$$1 . \det(A^{a_j}) = \det(A^{a_j}) = \det(A^{a_j}) . 1$$

We now show that 1 is in D. Let us define a tensor of mth rank E such that

$$E^{a_1 a_2 \cdots a_m} \; ; \; a_j = 1, 2, \ldots, n \quad \text{are given by}$$

$$E^{a_1 a_2 \cdots a_m} = +1 \quad \text{if} \quad a_1 = a_2 = \ldots = a_m$$
$$= 0 \quad \text{otherwise} \quad .$$

We note that $\det(E^{a_j}) = 1$ and so 1 is in D.

(iv) For $\det(A^{a_j})$ in D, there exists $\det(A^{a_j^{-1}})$ in D, such that

$$\det(A^{a_j})\det(A^{a_j^{-1}}) = 1 = \det(A^{a_j^{-1}})\det(A^{a_j}) \quad .$$

For this very condition we need that $\det(A^{a_j}) \neq 0$. Therefore A^{a_j} must be non-singular.

(v) Also we have

$$\det(A^{a_j})\det(B^{b_k}) = \det(B^{b_k})\det(A^{a_j})$$

Therefore, all non-zero determinants of a general tensor form an abelian group under matrix multiplication.