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DETERMINANT OF GENERAL TENSOR

by

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A generalized definition of a determinant of m th rank is proposed.

While working with tensors, it is sometimes necessary to introduce determinants of tensors of order higher than two. Let us see how the determinant of a second rank tensor (matrix) looks like

$$\det(T^{ab}) = \epsilon_{i_1 i_2 \dots i_n} T^{1i_1} T^{2i_2} \dots T^{ni_n} \quad (1)$$

where $a, b = 1, 2, \dots, n$ and T^{1i_1} are the components of T in the first row and i_1 -th column. $\epsilon_{i_1 i_2 \dots i_n}$ is the Levi-Civita tensor, which is of n th rank, and completely anti-symmetric. Since

$$\epsilon_{1 2 \dots n} = +1 \quad (2)$$

we can write (1) as

$$\det(T^{ab}) = \epsilon_{1 2 \dots n} \epsilon_{i_1 i_2 \dots i_n} T^{1i_1} T^{2i_2} \dots T^{ni_n} \quad (3)$$

This suggests that we can define the determinant of the m th rank tensor as the product of two terms

$$\det(T^{a_1 a_2 \dots a_m}) = (\text{product } \epsilon)(\text{product } T) \quad (4)$$

where

$$\begin{aligned} (\text{product } \epsilon) &= (\epsilon_{1 2 \dots n})(\epsilon_{i_1^{(1)} i_2^{(1)} \dots i_n^{(1)}}) \dots (\epsilon_{i_1^{(m)} \dots i_n^{(m)}}) \\ (\text{product } T) &= (T^{1i_1^{(1)} i_1^{(1)} \dots i_1^{(m)}}) \dots (T^{ni_n^{(1)} i_n^{(1)} \dots i_n^{(m)}}) \end{aligned}$$

$a_1, a_2, \dots, a_m = 1, 2, \dots, n$ and $i_1^{(p)}, p = 1, 2, \dots, m$ are the values that the indices a_1, a_2, \dots, a_m take. We take $i_1^{(1)} = 1, i_2^{(1)} = 2, \dots, i_n^{(1)} = n$. This can also be written in compact form

$$\det(T^{a_1 a_2 \dots a_m}) = \prod_{p=1}^m (\epsilon_{i_1^{(p)} i_2^{(p)} \dots i_n^{(p)}}) \prod_{q=1}^n (T^{q i_1^{(1)} i_1^{(2)} \dots i_1^{(m)}}) \dots (T^{q i_n^{(1)} i_n^{(2)} \dots i_n^{(m)}}) \quad (5)$$

Note that instead of rows and columns in a matrix we can obtain different 'arrays' by fixing each of the a_1, a_2, \dots, a_m one-by-one. It can be easily verified that

- (a) the determinant is zero if any 'array' consists of zeros;
- (b) the determinant is multiplied by a number A if any 'array' is multiplied by A ;
- (c) the determinant changes sign by the interchange of two adjacent 'arrays';
- (d) the determinant vanishes if two 'arrays' are identical.

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We now prove the following results.

Theorem 1: The determinant of an inner product is equal to the determinant of the outer product.

For a second rank tensor

$$\det(A^{ab}) = \epsilon_{a_1 a_2 \dots a_n} \epsilon_{b_1 b_2 \dots b_n} A^{a_1 b_1} A^{a_2 b_2} \dots A^{a_n b_n} \quad (6)$$

Now

$$\det(A^{ab}) \det(B^{cd}) = (\epsilon_{a_1 \dots a_n} \epsilon_{b_1 \dots b_n} \epsilon_{c_1 \dots c_n} \epsilon_{d_1 \dots d_n}) \times (A^{a_1 b_1} B^{c_1 d_1} \dots A^{a_n b_n} B^{c_n d_n})$$

If we replace c by b and sum over b we have

$$\det(A^{ab}) \det(B^{bd}) = [\epsilon_{a_1 \dots a_n} (\epsilon_{b_1 \dots b_n})^2 \epsilon_{d_1 \dots d_n}] \times [A^{a_1 b_1} B^{b_1 d_1} \dots A^{a_n b_n} B^{b_n d_n}]$$

$(\epsilon_{b_1 \dots b_n})^2 = +1$ when it is non-vanishing. In those cases we may define

$$A^{a_i b_i} B^{b_i d_i} = (AB)^{a_i d_i} \quad (7)$$

as the inner product of A and B . Therefore

$$\det(A^{ab}) \det(B^{bd}) = \det(AB^{ad}) \quad (8)$$

Note that we can change contravariant tensors to covariant (for their contraction) by multiplication with $\eta_{\mu\nu}$ having a determinant -1 . If we define

$$A^{a_i b_i} B^{c_i d_i} = (AB)^{a_i b_i c_i d_i} \quad (9)$$

as the outer product of tensors A and B , we notice that

$$\det(A^{ab}) \det(B^{cd}) = \det(AB^{abcd}) \quad (10)$$

We note that determinant is a number which does not depend on the choice of indices and so from equations (8) and (10) we have

$$\det(A^{ab}) \det(B^{bd}) = \det(A^{ab}) \det(B^{cd}) \quad (11)$$

and so we have

$$\det(AB^{abcd}) = \det(AB^{ab}) \quad (12)$$

Now we prove this theorem for a general tensor of rank n . Suppose it is true for a tensor of rank k ,

$$\det(T^{a_1 a_2 \dots a_k a_{k+1} a_{k+2}}) = \det(T^{a_1 a_2 \dots a_k}) \quad (13)$$

Now we take a new tensor whose components are given by C^{a_0} which is of first rank. Therefore according to equation (13)

$$\det(C^{a_0}) \det(T^{a_1 a_2 \dots a_k a_{k+1} a_{k+2}}) = \det(C^{a_0}) \det(T^{a_1 a_2 \dots a_k})$$

By definition (5) we see that the two sides are the determinants of tensors $(CT)^{a_0 a_1 \dots a_{k+2}}$ and $(CT)^{a_0 a_1 \dots a_k}$ and so

$$\det(CT^{a_0 a_1 \dots a_{k+2}}) = \det(CT^{a_0 a_1 \dots a_k}) \quad (14)$$

Therefore the assumption that the theorem is true for k implies that it is true for $k+1$ because on the right-hand side we have a tensor of rank $k+1$. We already proved that it is true when $k=2$. Hence it is true for all $n > 2$. For $n=1$ the result can be easily proved.

Theorem 2: The determinants of a general tensor form an abelian group under multiplication.

Let $\det(A^{a_1 a_2 \dots a_m})$ and $\det(B^{b_1 b_2 \dots b_n})$ be two elements in set D .

(i) Using theorem 1 we can show that

$$\det(A^{a_1 a_2 \dots a_m}) \det(B^{b_1 b_2 \dots b_n}) = \det(AB^{a_1 \dots a_m b_1 \dots b_n}) \in D$$

(ii) For all $\det(A^{a_j})$, $\det(B^{b_k})$ and $\det(C^{c_l})$ in D , $j = 1, 2, \dots, m$; $k = 1, 2, \dots, n$; $l = 1, 2, \dots, p$. Therefore

$$\det(A^{a_j}) [\det(B^{b_k}) \det(C^{c_l})] = \det(ABC^{a_j b_k c_l}) = [\det(A^{a_j}) \det(B^{b_k})] \det(C^{c_l})$$

by theorem 1.

(iii) There exists 1 in D such that

$$1 \cdot \det(A^{a_j}) = \det(A^{a_j}) = \det(A^{a_j}) \cdot 1$$

We now show that 1 is in D . Let us define a tensor of m th rank E such that

$$\begin{aligned} E^{a_1 a_2 \dots a_m} & ; a_j = 1, 2, \dots, n \quad \text{are given by} \\ E^{a_1 a_2 \dots a_m} & = +1 \quad \text{if } a_1 = a_2 = \dots = a_m \\ & = 0 \quad \text{otherwise} \end{aligned}$$

We note that $\det(E^{a_j}) = 1$ and so 1 is in D .

(iv) For $\det(A^{a_j})$ in D , there exists $\det(A^{a_j^{-1}})$ in D , such that

$$\det(A^{a_j}) \det(A^{a_j^{-1}}) = 1 = \det(A^{a_j^{-1}}) \det(A^{a_j})$$

For this very condition we need that $\det(A^{a_j}) \neq 0$. Therefore A^{a_j} must be non-singular.

(v) Also we have

$$\det(A^{a_j}) \det(B^{b_k}) = \det(B^{b_k}) \det(A^{a_j})$$

Therefore, all non-zero determinants of a general tensor form an abelian group under matrix multiplication.