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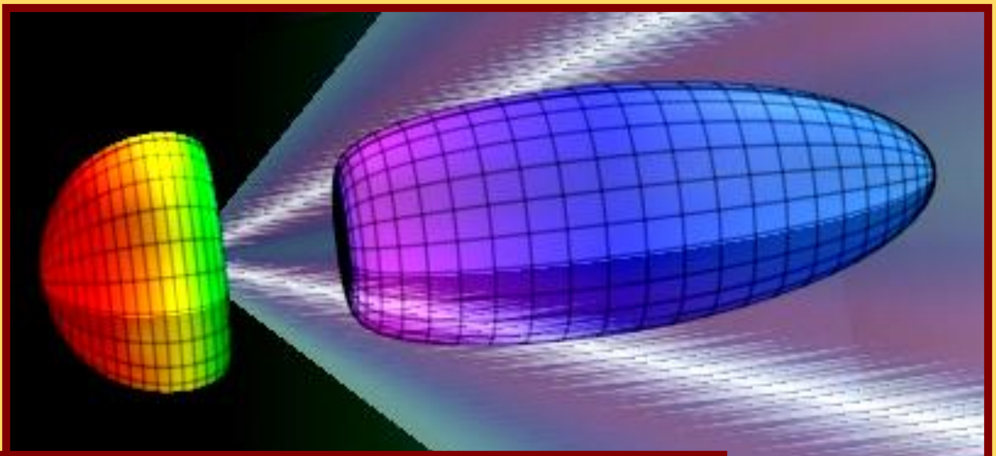
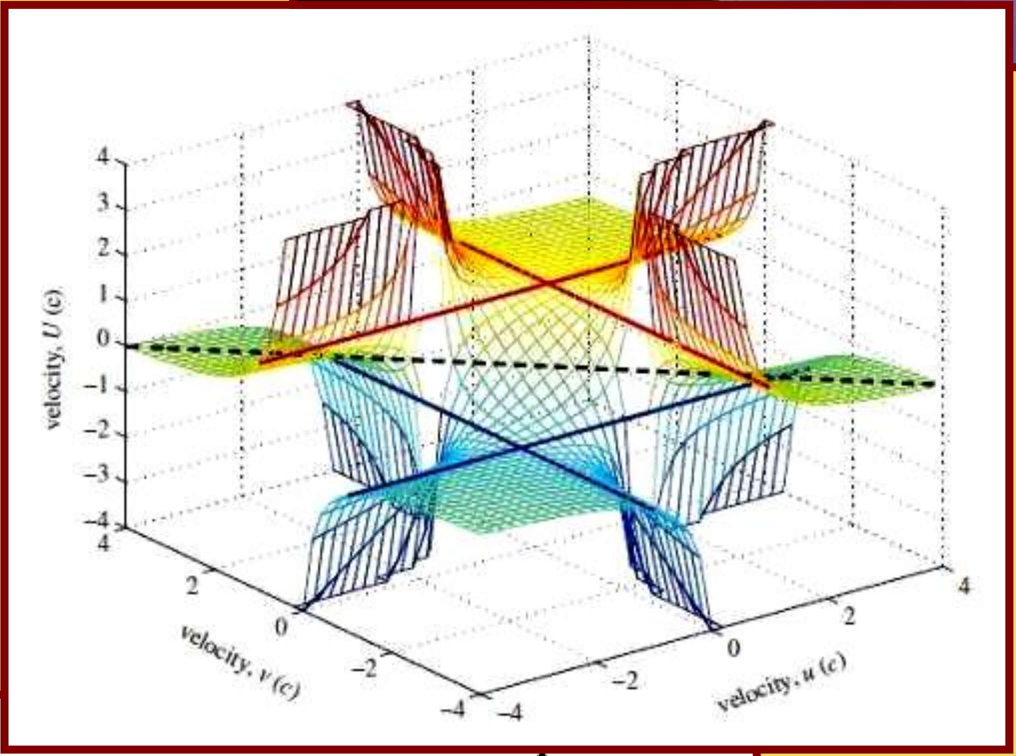
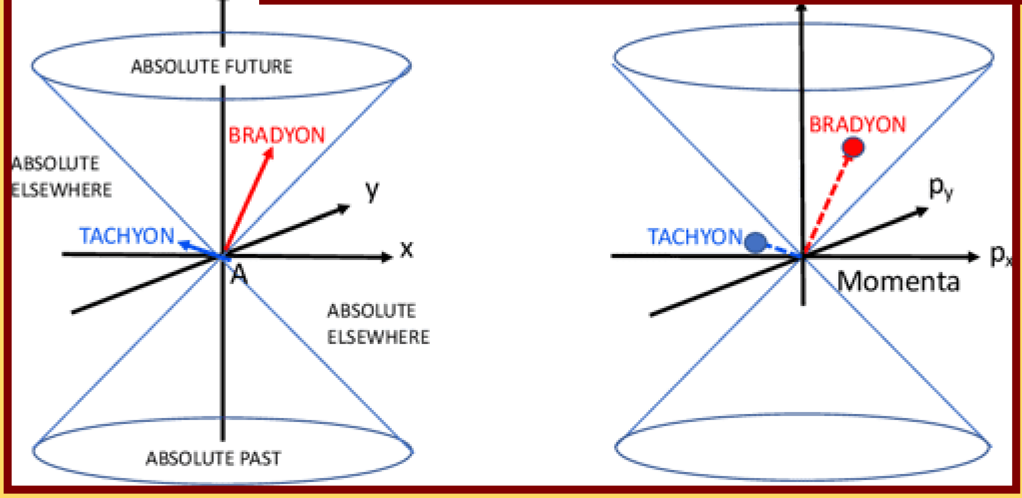
# Physical Significance and Properties of Relativistic-Effective-Mass Operator

The author introduced relativistic-effective-mass operator (REMO) while discussing the possibility of massive particles traveling with the velocity of light. This concept led to the modified expressions of special theory of relativity. Also, a very useful transformation for changing any luxon equation to the corresponding bradyon equation was developed. It is desirous to study the properties ⇒ ⇒ of REMO and its physical significance. REMO for tachyons is, also, discussed and a comparison for the cases of bradyon and tachyon is made. In this process, a unique derivation of Dirac equation is given using the concept of operators. This derivation, also, leads to the physical basis of different Dirac matrices. Equation of REMO is then derived using the same physical reasoning.







## Syed Arif Kamal

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# PHYSICAL SIGNIFICANCE AND PROPERTIES OF RELATIVISTIC EFFECTIVE MASS OPERATOR

**ARIF KAMAL***

**Department of Physics, University of Karachi, Karachi Pakistan**

## Abstract

The physical significance and properties of the relativistic effective mass operator (REMO) are discusesd. Relativistic effective mass operator for tachyons is defined and its properties are compared with REMO for bradyons.

## 1. INTRODUCTION

The author introduced relativistic effective mass operator (REMO) while discussing the possibility of massive particles travelling with the velocity of light [1]. This concept led to the modified expressions of special theory of relativity [2]. Also a very useful transformation for changing any luxon equation to the corresponding bradyon equation was developed.

It is desirous to study the properties of REMO and its physical significance. REMO for tachyons is also discussed and a comparison for the cases of bradyon and tachyon is made.

## 2. REMO FOR BRADYONS

The equation relating the energy E and momentum $p$ of a bradyon having proper mass $m_o$ can be written as

$$E^2 = p^2c^2 + m_o^2c^4 \tag{1}$$

Recalling that $E = mc^2$, the above equation can bere arranged as

$$p = (m^2 - m_o^2)^{\frac{1}{2}}\, c \tag{2}$$

From here we note that the bradyon can also be considered as a

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*Syed Arif Kamal, MSc, *summa cum laude*, BSc (Hons), *summa cum laude*, University of Karachi, Karachi, Pakistan • *paper mail* (*at the time of publication*): Associate Instructor, Department of Physics, Indiana University, Bloomington, Indiana 47405, United States • *e-mail*: profdrakamal@gmail.com • *homepage*: https://www.ngds-ku.org/kamal luxon with mass $(m^2 - m_\circ^2)^{\frac{1}{2}}$ . We call this as relativistic effective mass. The following equation is satisfied by such particles :

$$E_f^2 = p^2c^2 \tag{3}$$

where $E_f = (m^2 - m_\circ^2)^{\frac{1}{2}} c^2$ is the relativistic effective energy ; a detailed discussion has been given in ref. [1].

Introducing spin we can write the relativistic effective mass operator (REMO) as

$$M = \delta_1 m + \delta_2 m_\circ \tag{4}$$

where $\delta_1$ and $\delta_2$ are the matrices satisfying the conditions $\delta_1^2 = -\delta_2^2 = 1,\ \delta_1\delta_2 + \delta_2\delta_1 = 0.$

$M = \delta_1 m - \delta_2 m_\circ$ can also be taken as REMO with same conditions imposed on $\delta_1$ and $\delta_2$. For spin $\frac{1}{2}$ particles $\sigma_1$ and $i\sigma_2$, $\sigma_2$ and $i\sigma_3$, $\sigma_3$ and $i\sigma_1$ etc. can be taken as $\delta_1$ and $\delta_2$ where $\sigma$'s are Pauli spin matrices.

We now discuss some of the properties of REMO for bradyons. In doing so we shall take $\sigma_1$ and $i\sigma_2$ as $\delta_1$ and $\delta_2$. In this case $\delta_1{}^\dagger\delta_2 = 1$ and $\delta_2{}^\dagger\delta_2 = 1$ and so $\delta$'s are unitary. Further $\delta_1{}^\dagger = \delta_1$ *i.e.* $\delta_1$ is Hermitian but $\delta_2{}^\dagger = -\delta_2$ which shows that $\delta_2$ is anti-Hermitian.

(*i*) Eigenvalues : The eigenvalues come out to $+(m^2 - m_\circ^2)^{\frac{1}{2}}$ and $-(m^2 - m_\circ^2)^{\frac{1}{2}}$ .

(*ii*) Plane wave solutions : The plane wave solutions of $M\psi = (\delta_1 m + \delta_2 m_\circ)\ \psi$ where $\psi = \psi_j$ ; $j = 1,2$ and $\psi_j = u_j \exp{(i/\hbar)} (\vec{p}.\vec{r} - Et)$ can be written [1] as

$$u_{1\pm} = (m+m_\circ)c^2 (E_{f\pm})^{-1} A_\pm = E_{f\pm} (m-m_\circ)^{-1} c^{-2} A_\pm ;$$

$$u_{2\pm} = A_\pm \tag{5}$$

(*iii*) Probability density : The probability density is given by

$$|\psi|^2 = u_1^* u_1 + u_2^* u_2 = 2m(m-m_\circ)^{-1}\ |A_\pm|^2 \qquad (6)$$

Applying the condition $|\psi|^2 = 1$, we get

$$|A_\pm| = (m-m_\circ)^{\frac{1}{2}}\ (2m)^{-\frac{1}{2}} \qquad (7)$$

(*iv*) Normalized eigenvectors : The normalized eigenvectors are

$$\psi_+ = (2m)^{-\frac{1}{2}} \begin{bmatrix} (m+m_\circ)^{\frac{1}{2}} \\ (m-m_\circ)^{-\frac{1}{2}} \end{bmatrix} ;\ \psi_- = (2m)^{-\frac{1}{2}} \begin{bmatrix} (m+m_\circ)^{\frac{1}{2}} \\ -(m-m_\circ)^{-\frac{1}{2}} \end{bmatrix} \qquad (8a,b)$$

(*v*) Expectation value : For the solution (5) the expectation value comes out to expectation

$$\langle M \rangle = 2m(m+m_\circ)^{\frac{1}{2}}\ (m-m_\circ)^{-\frac{1}{2}} \left| A_\pm \right|^2 = (m^2-m_\circ^2)^{\frac{1}{2}} \qquad (9)$$

The last step is obtained using eq. (7)

(*vi*) Calculation of uncertainities : Using the relation of Robertson [3].

$$\Delta A\ \Delta B \geqslant \tfrac{1}{2}\,|\langle [A,B] \rangle| \qquad (10)$$

we calculate the uncerainities of $\delta_1$ and $\delta_2$, $\delta_1$ and M, $\delta_2$ and M.

(*a*) $\Delta\,\delta_1\ \Delta\,\delta_2 \geqslant \frac{1}{2}\,|\langle[\delta_1, \delta_2]\rangle| = \frac{1}{2}\,|\langle[\sigma_1, i\sigma_2]\rangle|$

On using eq. (7), we get

$$\Delta\,\delta_1\ \Delta\,\delta_2 \geqslant (2m_\circ)\,(m-m_\circ)^{-1} \left| A_\pm \right|^2 = m_\circ/m \qquad (11)$$

Therefore for all bradyons $(m_\circ \neq 0)$ $\delta_1$ and $\delta_2$ cannot be determined simultaneously.

(*b*) $\Delta\,\delta_1\ \Delta\,M \geqslant \frac{1}{2}\,|\langle[\delta_1, M]\rangle| = \frac{1}{2}\,|\langle[\sigma_1, \sigma_1 m + i\sigma_2 m_\circ]\rangle|$
$= \frac{1}{2}\,|\langle[\sigma_1, i\sigma_2 m_\circ]\rangle|$ or

$$\Delta\,\delta_1\ \Delta\,M \geqslant (2m_\circ^2)\ (m-m_\circ)^{-1} \left| A_\pm \right|^2 = m_\circ^2/m \qquad (12)$$

Therefore for all bradyons $(m_o \neq 0)$, $\delta_1$ and M cannot be simultaneously determined.

(*c*) $\Delta\,\delta_2\ \Delta\,M \geqslant \frac{1}{2}\,|\langle[\delta_2, M]\rangle| = |\langle[i\sigma_2, \sigma_1 m + i\sigma_2 m_o]\rangle|$
$= \frac{1}{2}\,|\langle[i\sigma_2, \sigma_1 m]\rangle|$

or $$\Delta \delta_2 \Delta \mathrm{M} \geqslant (2mm_o)(m-m_o)^{-1} \left| \mathrm{A}_{\pm} \right|^2 = m_o \qquad (13)$$

As the mass $m_o$ becomes higher and higher the uncertainity becomes greater and greater. Note that $\mathrm{M}^{\dagger} = \sigma m - i\sigma_2 m_o$ which is another form of REMO for bradyons.

## REMO FOR TACHYONS

Feinberg [4] gave the relation for the energy and momentum of tachyons (faster than light particles)

$$c^2p^2 - \mathrm{E}^2 = m_*^2c^4 \qquad (14)$$

where $m_o = im_*$ is the proper mass which is imaginary. $m_*$ is given the name 'mets mas$_*$' [5]. If $\mathrm{E} = mc^2$, $p = mv$ where

$\mathrm{m} = \mathrm{m}_o (1 - v^2/c^2)^{-\frac{1}{2}} = i\mathrm{m}_* (1 - v^2/c^2)^{-\frac{1}{2}} = \mathrm{m}_* (v^2/c^2 - 1)^{-\frac{1}{2}}$, we can write eq (14) as

$$p = (\mathrm{m}^2 + \mathrm{m}_*^2)^{\frac{1}{2}} c \qquad (15)$$

Both m and $\mathrm{m}_*$ are real and so the relativistic effective mass $(\mathrm{m}^2 + \mathrm{m}_*^2)^{\frac{1}{2}}$ is also real. In this case also the particle satisfies

$$\mathrm{E}_f^2 = c^2p^2 \qquad (16)$$

where $\mathrm{E}_f = (\mathrm{m}^2 + \mathrm{m}_*^2)^{\frac{1}{2}} c^2$ as relativistic effective energy.

Introducing spin, we can write the relativistic effective mass operator as

$$\mathrm{M} = \rho_1\mathrm{m} + \rho_2\mathrm{m}_* \qquad (17)$$

where $\rho_1^2 = 1 = \rho_2^2$, $\rho_1\rho_2 + \rho_2\rho_1 = 0$. For spin $\frac{1}{2}$ particles we can write $\mathrm{M} = \sigma_1\mathrm{m} + \sigma_2\mathrm{m}_*$ ; $\mathrm{M} = \sigma_1\mathrm{m} - \sigma_2\mathrm{m}_*$ (18a,b)

where $\sigma_1$ and $\sigma_2$ are Pauli matrices. We can also use $\sigma_2 m \pm \sigma_3 m_*$, $\sigma_3 m \pm \sigma_1 m_*$ etc. as relativistic effective mass operators. In the following discussion we shall use the expression of REMO given in eq. (18a).

Some of the properties of relative mass operator for tachyons are given below.

(*i*) Eigenvalues : The eigenvalues come out to be $+(m^2+m_*^2)^{\frac{1}{2}}$ and $-(m^2 + m_*^2)^{\frac{1}{2}}$ .

(*ii*) Plane wave solutions : The plane wave solutions of $M\psi = (\sigma_1 m + \sigma_2 m_*)\psi$ where $\psi = [\psi_j]$ ; $j = 1,2$ and $\psi_j = u_j \exp(i/\hbar)(\vec{p}.\vec{r} - Et)$ come out to be

$$u_{1\pm} = (m - im_*)c^2 (E_{f\pm})^{-1} B_{\pm} = E_{f\pm} (m + tm_*)^{-1} c^{-2} B_{\pm} \; ;$$

$$u_{2\pm} = B_{\pm} \tag{19a, b}$$

(*iii*) Probability density : The probability density is

$$|\psi|^2 = u_1^* u_1 + u_2^* u_2 = 2 |B_{\pm}|^2 \tag{20}$$

Applying the condition $|\psi|^2 = 1$ we get

$$|B_{\pm}| = (\tfrac{1}{2})^{\frac{1}{2}} \tag{21}$$

(*iv*) Normalized eigenvectors : The normalized eigenvectors corresponding to the positive and negative eigenvalues are

$$\psi_+ = (\tfrac{1}{2})^{\frac{1}{2}} (m^2 + m_*^2)^{-\frac{1}{2}} \begin{bmatrix} (m^2 + m_*^2)^{\frac{1}{2}} \\ (m + im_*) \end{bmatrix}$$

$$\psi_- = (\tfrac{1}{2})^{\frac{1}{2}} (m^2 + m_*^2)^{-\frac{1}{2}} \begin{bmatrix} (m^2 + m_*^2)^{\frac{1}{2}} \\ -(m + im_*) \end{bmatrix} \tag{22a, b}$$

(*v*) Expectation value : For the solution (19) the expectation value comes out to be expectation

$$<M> = 2(m^2 + m_*^2)^{\frac{1}{2}} |B_{\pm}|^2 = (m^2 + m_*^2)^{\frac{1}{2}} \tag{23a, b}$$

The last step is written because of eq. (21).

(*vi*) Calculation of uncertainities : Using the relation of Robertson (eq. 10) we calculate the uncertainities of $\sigma_1$ and $\sigma_2$, $\sigma_1$ and M, $\sigma_2$ and M.

(*a*) $\Delta\sigma_1 \Delta\sigma_2 > \frac{1}{2} | <[\sigma_1, \sigma_2]> |$

Therefore

$$\Delta\sigma_1 \, \Delta\sigma_2 \geq 0 \tag{24}$$

On using (19). This result shows that $\sigma_1$ and $\sigma_2$ can be determined simultaneously for tachyons.

(*b*) $\Delta\sigma_1 \, \Delta M \geq \frac{1}{2} \, | < [\sigma_1, M] > | = \frac{1}{2} \, | < [\sigma_1, \sigma_1 m + \sigma_2 m_*] > |$

Therefore

$$\Delta\sigma_1 \, \Delta M \geq 0 \tag{25}$$

Hence there is no uncertainity in the determination of M and $\sigma_1$ simultaneously.

(*c*) $\Delta\sigma_2 \, \Delta M > \frac{1}{2} \, | < [\sigma_2, M] > | = \frac{1}{2} \, | < [\sigma_2, \sigma_1 m + \sigma_2 m_*] > |$

Therefore

$$\Delta\sigma_2 \, \Delta M \geq 0 \tag{26}$$

Hence we conclude that there is no uncertainity in the determination of M for tachyons.

In the case of tachyons $M^\dagger = \sigma_1 m + \sigma_2 m_* = M$ and so M is Hermitian.

If we compare the results obtained for bradyons and for tachyons, we note that both possess real eigenvalues. But as regards uncertainities the results are remarkably different. For bradyons, uncertainities depend on $m_o$ in all cases and become zero only when $m_o = 0$. In the case of tachyons, the uncertainities are independent of $m_*$ and can be taken as zero. Further REMO for bradyons is not Hermitian whereas REMO for tachyons is Hermitian.

## PHYSICAL SIGNIFICANCE OF REMO

Dirac equation can be obtained using the concept of operators. In this case one is also led to the physical basis of different Dirac matrices [6]. If $\tau$ is the time in stationary frame of reference, we can write the rest mass energy operator as

$$m_o c^2 = i\hbar \, (\partial/\partial\tau)$$

Therefore

$$m_o c^2 = i\hbar \, (\partial t/\partial\tau)(\partial x/\partial t)(\partial/\partial x) + i\hbar \, (\partial t/\partial\tau)(\partial y/\partial t)(\partial/\partial y)$$
$$+ \, i\hbar \, (\partial t/\partial\tau)(\partial z/\partial t)(\partial/\partial z) + i\hbar \, (\partial t/\partial\tau)(\partial/\partial t)$$

This can be written as

$$i\hbar\,(\partial/\partial t) = -\,(\partial x/\partial t)\,i\hbar\,(\partial/\partial x) - (\partial y/\partial t)\,i\hbar\,(\partial/\partial y)$$
$$-\,(\partial z/\partial t)\,i\hbar\,(\partial/\partial z) + (\partial\tau/\partial t)\,i\hbar\,(\partial/\partial\tau)$$

$(\partial x/\partial t)$, $(\partial y/\partial t)$. $(\partial z/\partial t)$ are velocities and we can write $c\alpha_x$, $c\alpha_y$, $c\alpha_z$ for them. $-i\hbar\,(\partial/\partial x)$, $-i\hbar\,(\partial/\partial y)$, $i\hbar(\partial/\partial z)$ are momentum operators $p_x$, $p_y$, $p_z$ and $i\hbar\,(\partial/\partial\tau)$ is $m_0c^2$. Also $(\partial\tau/\partial t)$ is $(1-v^2/c^2)^{\frac{1}{2}}$ and we can write $\beta$ for it. Therefore we are led to the Dirac equation.

$$\mathrm{H} = c\alpha_x p_x + c\alpha_y p_y + c\alpha_z p_z + \beta m_0 c^2 \qquad (27)$$

To obtain the equation of REMO from physical reasoning, we start with the operator of rest mass energy, $m_0c^2$

$$m_0c^2 = i\hbar\,(\partial/\partial\tau) = i\hbar\,(\partial t/\partial\tau)\,(\partial/\partial t) + i\hbar\,(\partial r/\partial\tau)\,(\partial/\partial r)$$

Since $(\partial r/\partial\tau) = (\partial r/\partial t)\,(\partial t/\partial\tau)$ we can write

$$-\,i\hbar\,(\partial/\partial r) = (\partial t/\partial r)\,i\hbar\,(\partial/\partial t) - (\partial t/\partial r)\,(\partial\tau/\partial t)\,i\hbar\,(\partial/\partial\tau)$$

or using V instead of $(\partial/\partial r)$ we can write

$$-i\hbar\,\mathrm{V} = (\partial t/\partial r)\,i\hbar\,(\partial/\partial t) - (\partial\tau/\partial r)\,i\hbar\,(\partial/\partial\tau)$$

Now $-i\hbar\mathrm{V}$ is the operator for momentum which can be written as $Mc$ (definition of relativistic effective mass operator), $i\hbar(\partial/\partial t)$ is the energy $mc^2$, $i\hbar\,(\partial/\partial\tau)$ is the rest energy $m_0c^2$, $c(\partial t/\partial r)$ can be taken as $\delta_1$ and $c(\partial\tau/\partial r)$ as $\delta_2$. Therefore we obtain

$$Mc = \delta_1\,m\,c - \delta_2 m_0 c$$

or

$$\mathrm{M} = \delta_1 m - \delta_2 m_0 \qquad (28)$$

If we represent $c(\partial\tau/\partial r)$ as $-\delta_2$ we can write $\mathrm{M} = \delta_1 m + \delta_2 m_0$. A similar method gives the relativistic effective mass operator for tachyons.

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