سائنٹفک سوسائٹی پاکستان

خلاصے

شعبہ علوم طبیعی

انیسویں سالانہ کانفرنس

زیر اہتمام
جامعہ قالد اعظم اسلام آباد

۱۹۷۹ء

بریڈیان کے REMO کا ۲×۲ بیان

لیاقت علی ـ شیخ انصار حسین

عارف کمال اور زبان الکردی

شعبہ طبیعیات ، جامعہ کراچی

روشنی کی رفتار سے کم رفتار سے حرکت کرنے والے ذرات (بریڈیان Bradyon) کے لیے (REMO) اضافی موثر کمی ذارگر (Relativistic Effective Mass Operator) کا ۲×۲ بیان پیش کیا گیا ہے اور ان حالات کا جائزہ لیا گیا ہے جہاں اس کی ضرورت ہوتی ہے ۔

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4 x 4 REPRESENTATION OF REMO (RELATIVISTIC EFFECTIVE MASS OPERATOR) FOR BRADYONS*

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4 x 4 representation of relativistic effective mass operator (S. A. Kamal, J. Nat. Sci. Math. 20(1980)15) for bradyons (particles travelling with speeds less than the speed of light) is studied and the situations are considered where this representation is needed.

The concept of relativistic effective mass[1,2] has been very helpful in solving dynamical problems[2,3]. 2 x 2 representation of relativistic effective mass operator (REMO) has been described elsewhere[1,4,5]. 2 x 2 representation is useful only in parity violating systems[6,7]. In this paper a 4 x 4 representation of REMO and its properties are discussed[8]. Such a representation would be useful for the Dirac electron where we have four states[9].

Consider a bradyon having energy E, momentum p and proper (rest) mass m_o. We have

$$E^2 = p^2c^2 + m_o^2c^4 \quad (1)$$

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* Revised and updated English translation of the paper (in Urdu) presented during the Scientific Society of Pakistan's 19th Annual Science Conference, Quaid-i-Azam University, Islamabad (Pakistan), September 1979.

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Recalling that $E = mc^2$, we can write

$$mv = p = (m^2 - m_o^2)^{\frac{1}{2}} c \tag{2}$$

Eq. (2) shows that a bradyon of mass m moving with velocity v can be considered as a luxon of mass $(m^2 - m_o^2)^{\frac{1}{2}}$ moving with velocity c. $m_f = (m^2 - m_o^2)^{\frac{1}{2}}$ is called 'relativistic effective mass'[1,2].

Introducing spin we can write the relativistic effective mass operator as

$$M = \delta_1 m + \delta_2 m_o \tag{3}$$

where δ_1 and δ_2 are 4 x 4 matrices satisfying the conditions

$$\delta_1^2 = 1 = -\delta_2^2, \qquad \delta_1\delta_2 + \delta_2\delta_1 = 0 \tag{4}$$

Let us choose the following representation for δ_1 and δ_2.

$$\delta_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \tag{5}$$

where δ_1 and δ_2 are Pauli matrices and 0 is a 2 x 2 null matrix. Othe possible representations are

$$\delta_1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix} \tag{6a}$$

$$\delta_1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix} \tag{6b}$$

$$\delta_1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \tag{6c}$$

(6d) $\gamma_1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}$

(6e) $\gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}$

(7a) $\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}$

(7b) $\gamma_1 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}$

(7c) $\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}$

(7d) $\gamma_1 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}$

(7e) $\gamma_1 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}$

(7f) $\gamma_1 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}$

(8a) $\gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_3 \end{pmatrix}$

(8b) $\gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_3 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_1 \end{pmatrix}$

(8c) $\gamma_1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_3 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$

(9a) $\gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}$

(9b) $$\delta_1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$$

....

(10a) $$\delta_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}$$

(10b) $$\delta_1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$$

....

(11a) $$\delta_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

(11b) $$\delta_1 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

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Using δ_1 and δ_2 given in eq. (5), the following results are obtained:

(i) Eigenvalues: The eigenvalues come out to be

$$\pm (m^2 - m_o^2)^{\frac{1}{2}}, \qquad \pm (m^2 - m_o^2)^{\frac{1}{2}}$$

(ii) Plane wave solutions: Plane wave solutions for positive and negative energies can be written as

(12a,b) $$u_{1\pm} = A_\pm , \qquad u_{2\pm} = B_\pm$$

(12c,d) $$u_{3\pm} = (m + m_o)c^2(E_{f\pm})^{-1}B_\pm, \quad u_{4\pm} = (m - m_o)c^2(E_{f\pm})^{-1}A_\pm$$

where

(13) $$E_{f\pm} = \pm (m^2 - m_o^2)^{\frac{1}{2}}c^2$$

is the 'relativistic effective energy'.

(iii) Probability density: The probability density is given by

$$(14) \qquad |\psi|^2 = u_1^* u_1 + u_2^* u_2 + u_3^* u_3 + u_4^* u_4$$

Using eq. (12) we get

$$(15) \qquad |\psi|^2 = 2m(m + m_o)^{-1}|A_\pm|^2 + 2m(m - m_o)^{-1}|B_\pm|^2$$

Applying the condition $|\psi|^2 = 1$, we get the values of $|A_\pm|$ and $|B_\pm|$

$$(16a,b) \qquad |A_\pm| = \tfrac{1}{2}(1 + m_o/m)^{\frac{1}{2}}, \qquad |B_\pm| = \tfrac{1}{2}(1 - m_o/m)^{\frac{1}{2}}$$

(iv) Expected value: Using the values of $|A_\pm|$ and $|B_\pm|$ given above, the expected value

$$\langle M \rangle = 2m(m + m_o)^{-1} E_{f\pm} c^{-2} |A_\pm| + 2m(m - m_o)^{-1} E_{f\pm} c^{-2} |B_\pm|$$

can be written as

$$(17) \qquad \langle M \rangle = \pm (m^2 - m_o^2)^{\frac{1}{2}}$$

(v) Calculation of uncertainities: Using the relation of Robertson[10]

$$(18) \qquad \Delta A \Delta B \geqslant \tfrac{1}{2}|\langle [A,B] \rangle|$$

we calculate the uncertainities of δ_1 and δ_2, δ_1 and M, δ_2 and M.

$$(19a) \qquad \Delta\delta_1 \Delta\delta_2 \geqslant 2m_o(m + m_o)^{-1}|A_\pm|^2 + 2m_o(m - m_o)^{-1}|B_\pm|^2$$

Using eq. (16) we get

$$(19b) \qquad \Delta\delta_1 \Delta\delta_2 \geqslant m_o/m$$

It can be shown that[8]

(20) $$\Delta\delta_1 \Delta M \geqslant m_o \Delta\delta_1 \Delta\delta_2$$

(21) $$\Delta\delta_2 \Delta M \geqslant m \Delta\delta_1 \Delta\delta_2$$

Therefore

(22) $$\Delta\delta_1 \Delta M \geqslant m_o^2/m$$

(23) $$\Delta\delta_2 \Delta M \geqslant m_o$$

Therefore for all bradyons ($m_o \neq 0$), δ_1 and δ_2, δ_1 and M, δ_2 and M cannot be simultaneously determined.

We note that $M = \delta_1 m + \delta_2 m_o$ is not hermitian. This is due to the fact that $\delta_2^2 = -1$ and hence the eigenvalues of δ_2 are $\pm i$. Therefore δ_2 is not hermitian and hence M cannot be hermitian. There have been attempts to generalize the Dirac equation to any spin[11]. For such cases we have to use 8 x 8 or even higher representation of REMO. The operator

$$M = \delta_1 m + \delta_2 m_o$$

can be put in the form

(24) $$H = \delta_1 pc - \delta_1 \delta_2 m_o c^2$$

where we wrote p in place of Mc and H in place of mc^2. This Hamiltonian can be compared with any standard Hamiltonian.

For spin $\frac{1}{2}$ Dirac electrons, the Hamiltonian is

(25) $$H_D = \alpha_1 p_1 c + \alpha_2 p_2 c + \alpha_3 p_3 c + \beta m_o c^2$$

Therefore REMO can be written as

$$(26) \qquad H_f = Mc^2 = c(\theta_1 p_1 + \theta_2 p_2 + \theta_3 p_3 + \theta_4 p_4)$$

where

$$\theta_1 = \delta_1 \alpha_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

(I is a 2 x 2 unit matrix), $\theta_2 = \delta_1 \alpha_2 = i\alpha_3$, $\theta_3 = \delta_1 \alpha_3 = -i\alpha_2$,

$$\theta_4 = -i(\delta_1 \beta + \delta_2) = \begin{pmatrix} \sigma_2 - i\sigma_1 & 0 \\ 0 & \sigma_2 + i\sigma_1 \end{pmatrix}$$

$$p_4 = im_o c$$

H_f is the 'relativistic effective Hamiltonian operator' for Dirac electrons.

We have studied in detail 4 x 4 representation of δ matrices given by eq. (5). Other combinations of δ matrices given by equations (6-11) should also be studied to decide which representation is best suitable for a given range of velocity. The eigenvalues of 4 x 4 REMO are $\pm(m^2 - m_o^2)^{\frac{1}{2}}$, $\pm(m^2 - m_o^2)^{\frac{1}{2}}$. The positive value of relativistic effective mass is associated with the ordinary particles. However, the physical significance of negative value (recall that the particle is expressed as an equivalent luxon) has still to be studied.

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