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PHYSICS MAKES THE DEAF AND THE DUMB EQUATIONS OF MATHEMATICS TO SPEAK

KHURSHEED A. SIDDIQUI and SYED ARIF KAMAL *
Department of Physics, University of Karachi
Karachi 75270 (Pakistan)

Issac Assimov remarks: As for mathematics, that was, particularly, the tool of physicists, and as the research into first principles became more subtle and basic, it becomes nearly impossible to differentiate between the "pure mathematician" and the "theoretical physicist". There is, however, a difference between thinking of the above persons. A pure mathematician, mainly, works with an abstract set of axioms and tries to build a consistent theory based on these axioms. These axioms are a priori assumed to be correct. Theoretical physicists, also, work with hypotheses and conjectures, but their main criterion is the observable evidence. A model, which provides no verifiable test, is of little interest to physicists. Physicists change their assumptions and conjectures based on experimental evidence. Therefore, we notice that physics relates the abstract mathematical equations to down-to-earth problems and as such makes "the deaf and the dumb equations of mathematics to speak". A few mathematical equations and the physics behind them will be discussed.

INTRODUCTION

The deaf and the dumb equations of *mathematics* are made to speak through *physics* [1], which is the formulation of general laws applying, mainly, inductive logic. The journey from *mathematics* to *technolog*y could be considered as a journey from the abstract to the concrete, *mathematics* being in books, in the minds of philosophers, *physics* making contact with outside world, *technology* becoming the stage, where one

* Corresponding author, *homepage:* http://ngds-ku.org/kamal, *e-mail:* kamal(at the rate of)ngds-ku.org

enjoys the blessings. Let us look at a few examples to appreciate how physics gives language to the silent equations of mathematics.

DIRAC EQUATION

The time-dependent-Dirac equation may be written as[2]

(1) $$(i\mathbf{h}\frac{\partial}{\partial t} + i\mathbf{h}\boldsymbol{\alpha}.\nabla - m_0 c^2)y = 0$$

Since a_1, a_2, a_3, b are 4×4 matrices, plane-wave solutions of the form

(2) $$y_j(\boldsymbol{r},t) = u_j \exp i(\boldsymbol{k}.\boldsymbol{r} - wt); \quad j = 1,2,3,4$$

can be found. Substituting eq. (2) in eq. (1), we get four equations, which are homogeneous in u_j

(3*a*) $$(E - m_0 c^2)u_1 - cp_z u_3 - c(p_x - ip_y)u_4 = 0$$

(3*b*) $$(E - m_0 c^2)u_2 - c(p_x + ip_y)u_3 + cp_z u_4 = 0$$

(3*c*) $$(E + m_0 c^2)u_3 - cp_z u_1 - c(p_x - ip_y)u_2 = 0$$

(3*d*) $$(E + m_0 c^2)u_4 - c(p_x + ip_y)u_1 + cp_z u_2 = 0$$

u_j's have non-trivial solutions only if the determinant of the coefficient vanishes. This gives

(4) $$(E^2 - m_0^2 c^4 - c^2 p^2)^2 = 0$$

Therefore, we have

(5*a*, *b*) $$E_+ = \sqrt{c^2 p^2 + m_0^2 c^4}; \; E_- = -\sqrt{c^2 p^2 + m_0^2 c^4}$$

If the negative-energy states do exist, how can we keep the electron from tumbling into a negative-energy state, thus assuring the stability of the hydrogen-atom-ground state Dirac presented his relativistic equation[3] in 1928 and in 1930 he formulated the ***hole theory***[4]. He resolved this dilemma posed by the negative-energy solutions simply by filling up the negative-energy levels with electrons, in accordance with the Pauli exclusion principle. The vacuum state is then one with all negative-energy-electron levels filled and all positive-energy-levels empty.

It is possible for a negative-energy electron to absorb radiation and be exited into a positive-energy state. If this occurs, we observe an electron of energy E_+ and charge $-|e|$ and, in addition, a hole in the negative-energy sea. The hole suggests the absence of an electron of energy E_- and charge $+|e|$ (Fig. 1&2).

For a while, this remained a theoretical suggestion, only. In 1932 Anderson was investigating cosmic particles with cloud chambers divided in two by a lead barrier. One of Anderson's photograph showed a particle of starting characteristics to have been ejected from, the lead. From the extent of its curvature it seemed to have a mass equal to that of the electron, but it curved in the wrong direction. It was Dirac's positively-charged electron. Anderson named it the positron[5].

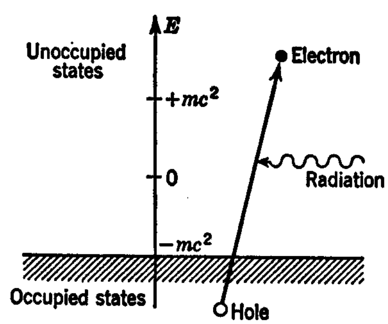

Fig. 1. Pair production: a negative-energy electron is excited to a positive-energy state by radiation (Courtesy: J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York, 1964, p. 65)

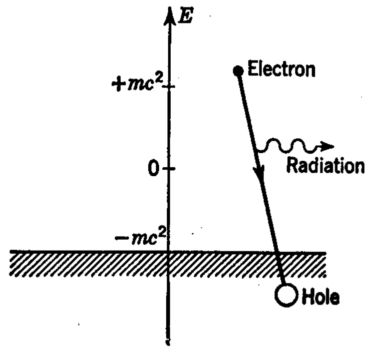

Fig. 2. Pair annihilation: a positive-energy electron falls into a negative-energy hole emitting radiation (Courtesy: J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York, 1964, p. 65)

Dirac and Anderson shared the 1933 Nobel prizes in physics, respectively, for these discoveries.

MAXWELL'S SECOND EQUATION

Maxwell's second equation states that the divergence of magnetic-flux density, always, vanishes, ***i. e.***,

$$\nabla \bullet \boldsymbol{B} = 0 \tag{5}$$

Since, the divergence is limiting case of flux per unit volume[6], the above equation shows that the lines of magnetic flux are, always, closed. We can not have an isolated magnetic pole. Remember, from our primary school science books that if we cut a magnet into 2 parts, each of the resulting part possesses a north and a south pole. This process continues no matter how small the parts are – each of the parts is a complete magnet possessing a north and a south pole. This observation gave rise to the domain theory of magnetism[7].

Dirac in 1931 proposed the idea that the mere existence of one magnetic monople in the universe would offer an explanation of the discrete nature of electric charge[8, 9]. The charge quantization everywhere requires a monopole of magnetic charge

$g = \frac{e}{2a_{FS}}$ (a_{FS} is the fine-structure constant). Monoples have, recently, become indispensable to many gauge theories, which endow them with a variety of extraordinary large masses [10-13].

The introduction of monoples needs a change in Maxwell's equations because $\nabla \bullet \boldsymbol{B} = 0$ is not consistent with the existence of a monople. The modified equations are written as[14, 15]

(7*a*, *b*)
$$\nabla \bullet \boldsymbol{D} = 4p r_e ;\ \nabla \times \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} + \frac{4p\boldsymbol{J}_e}{c}$$

(7*c*, *d*)
$$\nabla \bullet \boldsymbol{B} = 4p r_m ;\ -\nabla \times \boldsymbol{E} = \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} + \frac{4p\boldsymbol{J}_m}{c}$$

where $(r_e, \boldsymbol{J}_e)$ and $(r_m, \boldsymbol{J}_m)$ are electric and magnetic charge and current densities, respectively. The above equations (and all equations of electromagnetism appearing in this paper) are written in the Gaussian system. For conversions* to SI see ref. [14].

Particles endowed with both electric and magnetic charges are termed as dyons[15]. The electromagnetic force acting on a particle of electric charge ***e*** and magnetic charge ***g*** can be written as

(8)
$$\boldsymbol{F} = e(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}) + g(\boldsymbol{B} - \frac{1}{c}\boldsymbol{v} \times \boldsymbol{E})$$

For details on magnetic-monopole searches, the reader is referred to ref. [16].

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* A paper coauthored with Nasiruddin Bukhari (How to cope with different system of units?) was presented in a subsequent conference, specifically, dealing with conversion of electromagnetic equations from Gaussian System to SI and vice versa: <http://www.ngds-ku.org/pub/confabst1.htm#C25:>

CURL OF A VECTOR FIELD

The curl of a vector field is defined as the limiting case of line integral per unit area enclosed by the contour. The line integral represents a current. This is made explicit by considering Ampere's law

$$\oint \boldsymbol{H} \bullet \boldsymbol{dl} = \frac{4pi}{c} \tag{9}$$

where $\boldsymbol{H}$ is the magnetic-induction vector and $\boldsymbol{i}$ the electrical current (rate of flow of charge), enclosed in the loop under consideration. Faraday's law may be written as

$$\oint \boldsymbol{E} \bullet \boldsymbol{dl} = -\frac{1}{c}\frac{\partial f}{\partial t} \tag{10}$$

that is, the line integral of electric field is the negative of flux current (rate of change of flux)

The curl equation, thus, connects the rotation at a point to a curl. Consider the equation

$$\nabla \times (\nabla f) = 0 \tag{11}$$

that is, the curl of gradient is, identically, zero. The physics behind the above equation can be understood by noting that ∇f is the rate of change of a scalar field and $\nabla \times$ is the rotation at a point. Since the rate of change of a scalar field involves consideration of different layers, any attempt to rotate it at a single point would give a null result.

DIVERGENCE OF A VECTOR

Recall that divergence of a vector field is the limiting case of the flux per unit volume. If the divergence is zero at a point, there are no sources or sinks present at the point, *e. g.* $\nabla \bullet \boldsymbol{B} = 0$ implies existence of no isolated magnetic poles. However, if the divergence at a point is positive (negative), there is a source (sink) present at a point. For example, consider a point charge q located at the origin $(r = 0)$. The volume-charge density is given by

(12) $$r_Q(r) = qd(r)$$

where $d(r)$ is the Dirac delta function with the properties

(13) $$\int_{-\infty}^{+\infty} f(r)d(r - r_0)dr = f(r_0)$$

and

(14a) $$\int_a^b d(r - r_0)dr = 1 \qquad \text{if } r_0 \in (a,b)$$

(14b) $$= 0 \qquad \text{if } r_0 \notin (a,b)$$

From Gauss's law

(15) $$\nabla \bullet \boldsymbol{E} = 4pqd(r)$$

The quantity $\nabla.\boldsymbol{E} > 0$, if q is positive. The lines of force are coming out of the charged surface and this represents the presence of a source. The quantity $\nabla.\boldsymbol{E} < 0$, if q is negative. The lines of force are ending at the charged surface and this represents the presence of a sink. Once we learn to interpret the divergence as the flow of fields, we can understand the physics behind the vector equation

(16) $$\nabla \bullet (\nabla \times \boldsymbol{A}) = 0$$

that is, the divergence of a curl vanishes, identically. Curl represents rotation at a point, which cannot flow.

SYMMETRY AND CONSERVATION LAWS

Let us consider the following symmetries (Table 1), which we encounter in our everyday life:

Table 1. Symmetries encountered in physical systems

Language	*Mathematics*	*Conserved Quantity*
Isotropic (rotational)	$\partial y / \partial \Omega = 0$	Angular momentum (***L***)
Homogeneous (space translational)	$\nabla y = 0$	Linear momentum (***p***)
Stationary (time translational)	$\partial y / \partial t = 0$	Energy (€)

The quantity Ω is the solid angle.

Therefore, we note that symmetry is, always, related to a conservation law. Mathematically, if the lagrangian does not contain a particular coordinate q_i, the corresponding momentum is conserved, because

(17) $$\dot{p}_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial q_i} = 0$$

where $\dot{p}_i = \dfrac{dp_i}{dt}$. Conservation laws not related to any symmetry may be questionable, *e. g.*, conservation laws for baryon and lepton number, not related to any known symmetry were questioned in the SU(5) grand unification model proposed by Georgi and Glashow[17].

CONTINUITY EQUATION

Bruno Schmidt of the Karlsruhe Institute for the Didactics of Physics is developing a new curriculum for physics, which begins at the primary-school level and is intended to extend beyond high school into university studies.

Energy plays the primary role as part of a restructuring of physics, overall. In this restructuring *substance-like quantities* assume a fundamental place. A substance-like quantity is any physical quantity, which can be distributed in and flow through space, *e. g.*, energy (Є), charge (Q), amount of substance (n), linear momentum ($\boldsymbol{p}$), *etc.* These quantities are additive and they satisfy a continuity equation of the form

$$\frac{dr}{dt} + I = 0 \qquad (18)$$

where $\boldsymbol{I}$ is the current. For linear momentum

$$\frac{d\boldsymbol{p}}{dt} + \boldsymbol{I}_p = \frac{d\boldsymbol{p}}{dt} - \boldsymbol{F} = 0 \qquad (19)$$

Therefore, we see that $\boldsymbol{F}$ (force) may be interpreted as momentum current. For other substance-like quantities, we have (Table 2)

Table 2. Substance-like quantities and their corresponding currents

Substance-Like Quantity (r)	*Corresponding Current (I)*
Charge (Q)	Electrical
Entropy (S)	Thermal
Amount of Substance (n)	Chemical
Linear Momentum ($\boldsymbol{p}$)	Mechanical (translational)
Angular Momentum ($\boldsymbol{L}$)	Mechanical (rotational)

This approach lead to local cause or field theoretic approach to physics.

Structuring physics on the basis of the substance-like quantities, considerably, simplifies the description and unifies the rules and operations within many branches of physics and affects chemistry as well. At the same time this approach is easy to present at an elementary level.

SINGULARITIES

Singularities are points, where the mathematical equations give infinite or indeterminate values of the physical observables they describe. For example, the event horizon of a black hole, where the force acting on a body becomes infinite.

The singularities encountered in physics indicate there is need for a change in physics. Consider Rayleigh-Jeans law. It was derived using the basic principles of thermodynamics. The law gave results for longer wavelengths but the energy density approaches infinity for shorter wavelengths. This is termed as ***ultraviolet catastrophe***. Since there was no flaw in its derivation, Planck thought that there needs to be a change in physics to remove this singularity. This led to the birth of the quantum theory of radiation.

In the 1950's people doing calculations using quantum electrodynamics realized that the electron self-energy is coming out to be infinite, because of interaction with the virtual particles produced due to vacuum fluctuations. This led to the concept of renormalization in field theories. In the renormalized theories, the infinities cancel and we get a consistent description of the physical situation.

REFERENCES

1. I. Asimov, *The History of Physics* (Walker, New York, 1986), p. 3
2. L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill Kogakusha, Tokyo, 1988), p. 472
3. P. A. M. Dirac, *Proc. Roy. Soc. (London)*, **A 117** (1928) 610
4. P. A. M. Dirac, *Proc. Roy. Soc. (London)*, **A 126** (1930) 360
5. C. D. Anderson, *Phys. Rev.* **43** (1933) 491
6. P. Lorrain and D. R. Corson, *Electromagnetic Fields and Waves*, 2nd ed. (Freeman, San Francisco, 1970), p. 13
7. D. Halliday and R. Resnick, *Physics*, part 2, 3rd ed. (John Wiley, New Yoprk, 1976), p. 812

8. P. A. M. Dirac, *Proc. Roy. Soc. (London)*, **A 133** (1948) 60

9. P. A. M. Dirac, *Phys. Rev.*, **74** (1948) 817

10. J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (John Wiley, New York, 1975), p. 251

11. P. A. M. Dirac, *Directions in Physics*, ed. By H. Hora and J. R. Shepanski (John Wiley, New York, 1978), p. 39

12. C. R. Johnson, *Phys. Rev. D*, **24** (1981) 327

13. Z. Hovath and I. dalla, "Spontaneous compatification and monopoles in higher dimensions", *ICTP preprint* IC/78/37 (1978)

14. R. K. Wangness, Electroamgnetic Fields (John Wiley, New York, 1979), p. 414

15. H. Barcy and J. Kuber-Andre, *Int. J. Theor. Phys.*, **7** (1973) 39

16. Particle data Group, *Revs. Mod. Phys.*, **56 (No. 2, part II)** (1984) SI, p. S138

17. H. Georgi and S. L. Glashow, *Phys. Rev. Lett.* **32** (1974) 436